Nonlinear perturbations of linear elliptic systems at resonance

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Abstract

We consider a semilinear system

$$\Delta u + \lambda v + b_1(v) = f(x), \;\; x \in \Omega, \;\; u = 0 \;\text{ for } x \in \partial\Omega$$
$$\Delta v + \frac{\lambda_1^2}{\lambda} u + b_2(u) = g(x), \;\; x \in \Omega, \;\; v = 0 \;\text{ for } x \in \partial\Omega\,,$$

whose linear part is at resonance. Here $\lambda > 0$, the functions $b_1(t)$ and $b_2(t)$ are bounded and continuous. Assuming that $tb_i(t) > 0$ for all $t \in R$, $i = 1, 2$, and the first harmonics of $f(x)$ and $g(x)$ lie on a certain straight line, we prove existence of solutions. This extends a similar result for one equation, due to D.G. de Figueiredo and W.-M. Ni [5].

Key words: Elliptic system at resonance, existence of solutions.

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1 Introduction

Following publication of the classical paper of E.M. Landesman and A.C. Lazer [7], there has been an enormous interest in nonlinear perturbations of linear equations at resonance of the type

$$\Delta u + \lambda_1 u + b(u) = f(x), \;\; x \in \Omega, \;\; u = 0 \;\text{ for } x \in \partial\Omega\,, \tag{1.1}$$

where Ω is a bounded smooth domain in R^n, and λ_1 is the principal eigenvalue of the Laplacian $-\Delta$ on Ω, with zero at the boundary condition (we shall denote by $\phi_1(x)$ the corresponding eigenfunction). Early contributions included the other classics, A. Ambrosetti and G. Prodi [1] and M.S. Berger and E. Podolak [3], see a nice presentation in the book of A. Ambrosetti and G. Prodi [2]. Recently, the present author [6] has suggested a unified approach to these results. The function $b(u)$ is usually assumed to be bounded and continuous, and the famous E.M. Landesman and A.C. Lazer [7] conditions required that it had limits at $\pm\infty$. In an elegant paper, D.G. de Figueiredo and W.-M. Ni [5] proved existence of solutions assuming that $ub(u) > 0$ for all $u \in R$, and the forcing term $f(x)$ has zero first harmonic, i.e., $\int_\Omega f(x)\phi_1(x)\,dx = 0$. Their proof involved establishment of an a priori estimate, which was remarkable because such estimates usually require some conditions on $b(u)$ at infinity.

In this note we extend the result of D.G. de Figueiredo and W.-M. Ni [5] to a system of two equations. The system

$$\begin{aligned} &\Delta u + \lambda v + b_1(v) = f(x), \;\; x \in \Omega, \;\; u = 0 \;\text{ for } x \in \partial\Omega \\ &\Delta v + \tfrac{\lambda_1^2}{\lambda} u + b_2(u) = g(x), \;\; x \in \Omega, \;\; v = 0 \;\text{ for } x \in \partial\Omega\,, \end{aligned} \tag{1.2}$$

with any $\lambda > 0$ can be seen as the case of resonance at the principal eigenvalue, similarly to (1.1). Similarly to [5], we assume that $tb_i(t) > 0$ for all $t \in R$, $i = 1, 2$. We prove existence of solutions, provided that the first harmonics of $f(x)$ and $g(x)$ lie on a certain straight line. There is a considerable interest in systems of this type, see e.g., the recent surveys of D.G. de Figueiredo [4] and B. Ruf [9].

2 Existence of solutions

On a smooth domain $\Omega \subset R^n$, we consider a weakly coupled linear system

$$\begin{aligned} &\Delta u + \lambda v = f(x), \;\; x \in \Omega, \;\; u = 0 \;\text{ for } x \in \partial\Omega \\ &\Delta v + \bar\lambda u = g(x), \;\; x \in \Omega, \;\; v = 0 \;\text{ for } x \in \partial\Omega\,, \end{aligned} \tag{2.1}$$

with given functions $f(x)$ and $g(x)$, and parameters λ and $\bar\lambda$. The following proposition identifies the set of non-resonant parameters λ and $\bar\lambda$. We denote by λ_n the eigenvalues of $-\Delta$ on Ω, which vanish at the boundary, and by $\phi_n(x)$ the corresponding eigenfunctions.

Proposition 1 *Assume that $\lambda\bar\lambda \neq \lambda_n^2$ for all $n \geq 1$. Then for any pair $(f(x), g(x)) \in L^2(\Omega) \times L^2(\Omega)$ there exists a unique solution $(u(x), v(x)) \in \left(W^{2,2}(\Omega) \times W^{2,2}(\Omega)\right)^2$.*

Proof: Existence of solution in $L^2(\Omega) \times L^2(\Omega)$ follows by using the Fourier series in $\phi_n(x)$, written for u,v,f, and g, and then the standard elliptic estimates provide the extra regularity of solution. $\diamondsuit$

The *resonance* case is when $\lambda\bar{\lambda} = \lambda_n^2$. We shall consider the *principal* resonance case $\lambda\bar{\lambda} = \lambda_1^2$, i.e., $\bar{\lambda} = \frac{\lambda_1^2}{\lambda}$. We shall prove solvability for the system

$$\begin{aligned} \Delta u + \lambda v + b_1(v) &= f(x), \;\; x \in \Omega, \;\; u = 0 \;\text{ for } x \in \partial\Omega \\ \Delta v + \tfrac{\lambda_1^2}{\lambda} u + b_2(u) &= g(x), \;\; x \in \Omega, \;\; v = 0 \;\text{ for } x \in \partial\Omega , \end{aligned} \tag{2.2}$$

with given functions $f(x)$, $g(x) \in L^2(\Omega)$, and a constant $\lambda > 0$. The following is a system analog of the result of D.G. de Figueiredo and W.-M. Ni [5]. We denote $\phi_1^\perp = \{f \in L^2(\Omega) \, : \, \int_\Omega f\phi_1 \, dx = 0\}$.

Theorem 2.1 *Assume that $b_1(t)$ and $b_2(t)$ are bounded and continuous functions, such that*

$$tb_i(t) > 0 \quad \text{for all } t \in R, \; i = 1, 2 \, . \tag{2.3}$$

Decompose $f(x) = \mu_1\phi_1(x) + e_1(x)$, $g(x) = \nu_1\phi_1(x) + e_2(x)$, with $e_1(x)$, $e_2(x) \in \phi_1^\perp$. Then the system (2.2) is solvable for any (μ_1, ν_1) satisfying

$$\lambda_1\mu_1 + \lambda\nu_1 = 0 \tag{2.4}$$

with u, $v \in W_0^{1,2}(\Omega) \cap W^{2,p}(\Omega)$, for all $p > 2$.

The proof will be based on the following lemmas. The first one follows immediately by considering Fourier series in $\phi_n(x)$.

Lemma 2.1 *The solution set of the linear system*

$$\begin{aligned} \Delta u + \lambda v &= 0, \;\; x \in \Omega, \;\; u = 0 \;\text{ for } x \in \partial\Omega \\ \Delta v + \tfrac{\lambda_1^2}{\lambda} u &= 0, \;\; x \in \Omega, \;\; v = 0 \;\text{ for } x \in \partial\Omega \end{aligned} \tag{2.5}$$

is $(u, v) = c(\phi_1, \frac{\lambda_1}{\lambda}\phi_1)$, where c is an arbitrary constant. In particular, the only solution of (2.5) in $\phi_1^\perp \times \phi_1^\perp$ is $(0, 0)$.

Lemma 2.2 *Let U, $V \in \phi_1^\perp$ be solutions of*

$$\begin{aligned} \Delta U + \lambda V &= f(x), \;\; x \in \Omega, \;\; u = 0 \;\text{ for } x \in \partial\Omega \\ \Delta V + \tfrac{\lambda_1^2}{\lambda} U &= g(x), \;\; x \in \Omega, \;\; v = 0 \;\text{ for } x \in \partial\Omega , \end{aligned} \tag{2.6}$$

with $f(x)$, $g(x) \in L^\infty(\Omega)$. Then for any $p > 1$ one can find a constant $c > 0$, such that

$$||U||_{W^{2,p}(\Omega)} + ||V||_{W^{2,p}(\Omega)} \leq c(||f||_{L^p(\Omega)} + ||g||_{L^p(\Omega)}) \, . \tag{2.7}$$

By the Sobolev imbedding this implies that $||U||_{L^\infty(\Omega)} + ||V||_{L^\infty(\Omega)} \leq c_1$, for some constant $c_1 > 0$.

Proof: Standard elliptic estimates imply that

$$||U||_{W^{2,p}} + ||V||_{W^{2,p}} \leq c(||f||_{L^p} + ||g||_{L^p} + ||U||_{L^p} + ||V||_{L^p}) .$$

The estimate (2.7) will follow, once we prove that

$$||U||_{L^p} + ||V||_{L^p} \leq c(||f||_{L^p} + ||g||_{L^p}) . \tag{2.8}$$

Assume for definiteness that $||f||_{L^p} \geq ||g||_{L^p}$. Dividing both equations in (2.7) by the same constant $||f||_{L^p}$, and redefining U and V, we may assume that $||f||_{L^p} = 1$ and $|g||_{L^p} \leq 1$. Assuming that the estimate (2.8) is not possible with any constant c, we could find a sequence $\{f_n, g_n\}$, with $||f_n||_{L^p} = 1$ and $|g_n||_{L^p} \leq 1$, and the corresponding solutions of (2.6) $\{U_n, V_n\} \in \phi_1^\perp \times \phi_1^\perp$, so that

$$||U_n||_{L^p} + ||V_n||_{L^p} \geq n(1 + ||g_n||_{L^p}) .$$

In particular, $||U_n||_{L^p} + ||V_n||_{L^p} \to \infty$, as $n \to \infty$. Define $u_n = \dfrac{U_n}{||U_n||_{L^p} + ||V_n||_{L^p}}$ and $v_n = \dfrac{V_n}{||U_n||_{L^p} + ||V_n||_{L^p}}$. They satisfy

$$\begin{aligned} \Delta u_n + \lambda v_n &= \tfrac{f_n(x)}{||U_n||_{L^p} + ||V_n||_{L^p}} \\ \Delta v_n + \tfrac{\lambda_1^2}{\lambda} u_n &= \tfrac{g_n(x)}{||U_n||_{L^p} + ||V_n||_{L^p}} . \end{aligned} \tag{2.9}$$

Since $||u_n||_{L^p} < 1$, $||v_n||_{L^p} < 1$, we get uniform in n bounds for $||u_n||_{W^{2,p}}$ and $||v_n||_{W^{2,p}}$. In a standard way, along a subsequence $\{u_n, v_n\} \to (u, v) \in \phi_1^\perp \times \phi_1^\perp$, with (u, v) solving (2.5). Hence $u = v = 0$ by Lemma 2.2, but $||u + v||_{L^p} = 1$, a contradiction. $\diamondsuit$

The following lemma provides the crucial a priori estimate. As mentioned in D.G. de Figueiredo and W.-M. Ni, it is remarkable that this estimate does not require any conditions on $b_i(t)$ at infinity (which are usually needed to get a priori estimates).

Lemma 2.3 *In the conditions of the Theorem 2.1, there is a constant $c > 0$, so that any solution of (2.2) satisfies*

$$||u||_{L^2(\Omega)} + ||v||_{L^2(\Omega)} \leq c .$$

Proof: Decompose $u(x) = \xi_1 \phi_1(x) + U(x)$, $v(x) = \eta_1 \phi_1(x) + V(x)$, with $U(x)$, $V(x) \in \phi_1^\perp$. The system (2.2) becomes

$$\begin{aligned} &\Delta U + \lambda V + (-\lambda_1 \xi_1 + \lambda \eta_1)\phi_1 + b_1(\eta_1 \phi_1(x) + V(x)) = \mu_1 \phi_1(x) + e_1(x), \\ &\Delta U + \tfrac{\lambda_1^2}{\lambda} U + \tfrac{\lambda_1}{\lambda}(\lambda_1 \xi_1 - \lambda \eta_1)\phi_1 + b_2(\xi_1 \phi_1(x) + U(x)) = \nu_1 \phi_1(x) + e_2(x) . \end{aligned} \tag{2.10}$$

We claim that

$$(2.11) \qquad |-\lambda_1\xi_1 + \lambda\eta_1| \leq c\,,$$

for some constant $c > 0$. Indeed, multiply the first equation in (2.10) by ϕ_1, and integrate over Ω. Since $\int_\Omega \Delta U \phi_1\,dx = \int_\Omega V\phi_1\,dx = 0$, while b_1 is a bounded function, the claim follows. By Lemma 2.2, it follows that

$$(2.12) \qquad ||U||_{C^1(\Omega)} + ||V||_{C^1(\Omega)} \leq c_1\,,$$

for some constant $c_1 > 0$.

To complete the proof, we need an a priori estimate of the first harmonics ξ_1 and η_1. By (2.11), if either one of ξ_1 and η_1 is large and positive (negative), so is the other one. Assume for definiteness that ξ_1 and η_1 are both negative, and large in absolute value. Multiply the first equation in (2.10) by $\lambda_1\phi_1$, the second one by $\lambda\phi_1$, integrate over Ω, and add the results. We may assume that $\int_\Omega \phi_1^2\,dx = 1$. By our condition (2.4)

$$0 = \lambda_1\mu_1 + \lambda\nu_1 = \int_\Omega \left[\lambda_1 b_1(\eta_1\phi_1(x) + V(x)) + \lambda b_2(\xi_1\phi_1(x) + U(x))\right]\phi_1(x)\,dx\,.$$

We claim that the integral on the right is negative, which gives us a contradiction. Indeed, by (2.12), $\eta_1\phi_1(x) + V(x) < 0$ and $\xi_1\phi_1(x) + U(x) < 0$ over Ω, and then, by condition (2.3), the functions b_1 and b_2 are negative. $\diamondsuit$

Proof of the Theorem 2.1 Letting $w = (u, v)$, we rewrite the system (2.2) in the operator form

$$w = T(w)\,,$$

where $T(w) = \left(\Delta^{-1}\left(-\lambda v - b_1(v) + \mu_1\phi_1 + e_1\right), \Delta^{-1}\left(-\frac{\lambda_1^2}{\lambda}u - b_2(u) + \nu_1\phi_1 + e_2\right)\right)$. T is a compact map $L^2(\Omega)\times L^2(\Omega) \to L^2(\Omega)\times L^2(\Omega)$. We define $\mathbf{L}^2 = L^2(\Omega)\times L^2(\Omega)$, with the norm $||w||^2_{\mathbf{L}^2} = ||u||^2_{L^2(\Omega)} + ||v||^2_{L^2(\Omega)}$. Following D.G. de Figueiredo and W.-M. Ni [5], we consider the operator

$$T_k(w) = \frac{1}{k+1}T(w) - \frac{k}{k+1}T(-w), \;\; 0 \leq k \leq 1\,,$$

which is compact for all k, $T_0 = T$, and T_1 is an odd operator. It is known, see e.g., L. Nirenberg [8], that the Leray-Schauder degree

$$deg(I - T_1, B_R, 0) \neq 0$$

for any ball $B_R = \{w \in \mathbf{L}^2 \,:\, ||w||_{\mathbf{L}^2} \leq R\}$. We claim that there is an R such that

$$w - T_k(w) \neq 0, \;\; \text{for} \;\; ||w||_{\mathbf{L}^2} = R, \;\; 0 \leq k \leq 1\,.$$

Then by the homotopy invariance of the degree, $deg(I-T, B_R, 0) \neq 0$, which implies that the system (2.2) has a solution. To prove the claim, we need a uniform in k a priori bound for

$$w - T_k(w) = 0\,,$$

which is equivalent to

$$\begin{aligned} \Delta u + \lambda v + \tfrac{1}{k+1} b_1(v) - \tfrac{k}{k+1} b_1(-v) &= \tfrac{1-k}{1+k}\left(\mu_1 \phi_1 + e_1\right) \\ \Delta v + \tfrac{\lambda_1^2}{\lambda} u + \tfrac{1}{k+1} b_2(u) - \tfrac{k}{k+1} b_2(-u) &= \tfrac{1-k}{1+k}\left(\nu_1 \phi_1 + e_2\right)\,. \end{aligned} \tag{2.13}$$

Clearly, the condition (2.4) on the first harmonics is satisfied for all k. Letting $b_i^k(t) = \frac{1}{k+1} b_i(t) - \frac{k}{k+1} b_i(-t)$, $i = 1, 2$, we see that these functions are uniformly bounded in k, and satisfy the condition (2.3). By Lemma 2.3, we conclude a uniform in k a priori bound for solutions of (2.13), completing the proof. $\diamondsuit$

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