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CURVES OF EQUIHARMONIC SOLUTIONS, AND RANGES OF NONLINEAR EQUATIONS

Philip Korman
Department of Mathematical Sciences, University of Cincinnati
Cincinnati Ohio 45221-0025

(Submitted by: Reza Aftabizadeh)

Abstract. We consider the semilinear Dirichlet problem

$$\Delta u + kg(u) = \mu_1\varphi_1 + \ldots + \mu_n\varphi_n + e(x) \ \text{ for } x \in U, \ \ u = 0 \ \text{ on } \partial U,$$

where φ_k is the k-th eigenfunction of the Laplacian on U and $e(x) \perp \varphi_k$, $k = 1, \ldots, n$. We write the solution in the form $u(x) = \sum_{i=1}^n \xi_i\varphi_i + U_\xi(x)$, with $U_\xi \perp \varphi_k$, $k = 1, \ldots, n$. Starting with $k = 0$, when the problem is linear, we continue the solution in k by keeping $\xi = (\xi_1, \ldots, \xi_n)$ fixed, but allowing $\mu = (\mu_1, \ldots, \mu_n)$ to vary. We then study the map $\xi \to \mu$, which provides existence and multiplicity results for the above problem.

1. Introduction

We study existence and multiplicity of solutions for a semilinear problem

$$\begin{aligned} &\Delta u + kg(u) = f(x) \ \text{ for } x \in U, \\ &u = 0 \ \text{ on } \partial U \end{aligned} \tag{1.1}$$

on a smooth domain $U \subset R^m$. Here the functions $f(x) \in L^2(U)$ and $g(u) \in C^2(R)$ are given and k is a parameter. We shall approach this problem by continuation in k.

When $k = 0$ the problem is linear. It has a unique solution, as one sees by using Fourier series of the form $u(x) = \sum_{k=1}^\infty u_k\varphi_k$, where φ_k is the k-th eigenfunction of the Laplacian on U, and λ_k is the corresponding eigenvalue. We now continue in k, looking for a solution pair (k, u), or $u = u(x, k)$. At a generic point (k, u) the implicit function theorem applies, allowing the continuation in k. These are *regular* points, where the corresponding linearized problem has only the trivial solution. So until a *singular* point is encountered, we have a solution curve $u = u(x, k)$. At a singular point

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practically anything imaginable might happen (as one can see even on two-dimensional examples, dealing with the solution set of $h(k,u) = 0$, where h is some function of $k \in R$ and $u \in R$). At some critical points the M.G. Crandall and P.H. Rabinowitz theorem [6] applies, giving us a curve of solutions through a critical point. But even in this favorable situation there is a possibility that a solution curve will "turn back" in k.

So what is the way forward in k, which can take us through any critical point? If a solution $u(x)$ is given by a Fourier series $u(x) = \sum_{k=1}^{\infty} u_k \varphi_k$, we call $U_n = (u_1, u_2, \dots, u_n)$ the *n-signature* of the solution, or just signature for short. We write $f(x)$ by its Fourier series, and rewrite the problem (1.1) as

$$\begin{aligned} &\Delta u + kg(u) = \mu_1 \varphi_1 + \ldots + \mu_n \varphi_n + e(x) \quad \text{for } x \in U, \\ &u = 0 \quad \text{on } \partial U \end{aligned} \tag{1.2}$$

with $e(x)$ the projection of $f(x)$ onto the orthogonal complement to $\varphi_1, \dots, \varphi_n$. Let us now constrain ourselves to holding the signature U_n fixed (when continuing in k), and in return allow for $\mu_1, \dots, \mu_n$ to vary; i.e., we are looking for $(u, \mu_1, \dots, \mu_n)$ as a function of k, with U_n fixed. It turns out that we can continue forward in k this way, so long as

$$k \max_{u \in R} g'(u) < \lambda_{n+1}.$$

So suppose this condition holds, and we wish to solve the problem (1.1) at some $k = k_0$. We travel in k from $k = 0$ to $k = k_0$ on a curve of fixed signature $U_n = (u_1, u_2, \dots, u_n)$, obtaining a solution of (1.2). The right-hand side of (1.2) has the first n harmonics different from the ones we want in (1.1). We now vary U_n. The question is: can we choose U_n to obtain the desired $\mu_1, \dots, \mu_n$, and if so, in how many ways? This corresponds to the existence and multiplicity questions for the original problem (1.1). The classical results of E.M. Landesman and A.C. Lazer [12], A. Ambrosetti and G. Prodi [2], M. S. Berger and E. Podolak [4], as well as well-known papers of H. Amann and P. Hess [1] and D.G. de Figueiredo and W.-M. Ni [8] dealt with these questions when $n = 1$. We are able to obtain extensions of some of these results, and largely recover the others. We show that *all* solutions of the problem (1.2) (when $n = 1$) lie on a unique solution curve $\mu_1 = \mu_1(\xi_1)$, and this curve determines multiplicity of solutions. Thus, a two-dimensional curve gives a faithful representation of the solution set of (1.2). All solutions of the problem (1.2) can be numerically computed through two continuations: first in k, and then in ξ_1, and both of these continuations do not encounter turns or any other singularities.

We also study the ranges in the $n > 1$ case. We obtain some existence results, covering the case of resonance at higher eigenvalues. The advantage of our approach is that we have concrete solution curves in hand, when discussing ranges of nonlinear equations. Our approach can be seen as a dynamical version of the classical Liapunov-Schmidt procedure. We do not seek to solve the equation off the kernel in one step by applying an implicit function theorem, but instead perform continuation. The inverse (or control) problem that we had solved in the process: given U_n find $\mu_1, \dots, \mu_n$ (with $e(x)$ fixed), appears to be of independent interest. We apply our results to the question of symmetry breaking, considered in a recent paper by F. Pacella and P. N. Srikanth [14]. We extend that result, obtaining multiple symmetry breaking solutions.

Our approach is competitive for numerical computations. It is easy to implement continuation in parameters, if one is guaranteed that solutions do not turn back when parameters are varied. We performed such computations in our previous paper [11].

2. Preliminary results

Recall that on a smooth domain $U \subset R^m$ the eigenvalue problem

$$\Delta u + \lambda u = 0 \text{ on } U, \quad u = 0 \text{ on } \partial U$$

has an infinite sequence of eigenvalues $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots \to \infty$, where we repeat each eigenvalue according to its multiplicity, and the corresponding eigenfunctions we denote φ_k. These eigenfunctions φ_k form an orthogonal basis of $L^2(U)$; i.e., any $f(x) \in L^2(U)$ can be written as $f(x) = \sum_{k=1}^{\infty} a_k \varphi_k$, with the series convergent in $L^2(U)$, see e.g. L. Evans [7]. We may normalize $||\varphi_k||_{L^2(U)} = 1$ for all k.

Lemma 2.1. *Assume that $u(x) \in L^2(U)$, and $u(x) = \sum_{k=n+1}^{\infty} a_k \varphi_k$. Then*

$$\int_U |\nabla u|^2 \, dx \ge \lambda_{n+1} \int_U u^2 \, dx.$$

Proof. Since $u(x)$ is orthogonal to $\varphi_1, \ldots, \varphi_n$, the proof follows by the variational characterization of λ_{n+1}. □

Any $f(x) \in L^2(U)$ can be decomposed as $f(x) = \mu_1\varphi_1 + \ldots + \mu_n\varphi_n + e(x)$, with $e(x) = \sum_{k=n+1}^{\infty} \mu_k \varphi_k$ orthogonal to $\varphi_1, \ldots, \varphi_n$. We call the vector $\mu = (\mu_1, \ldots, \mu_n)$ the $n-$*signature* of $f(x)$, or *signature*, for short. We consider a boundary-value problem

$$\Delta u + kg(u) = \mu_1\varphi_1 + \ldots + \mu_n\varphi_n + e(x) \text{ for } x \in U, \tag{2.1}$$

$$u = 0 \text{ on } \partial U.$$

Here $k > 0$ is a constant, and $g(u) \in C^2(R)$ is assumed to satisfy

$$|g'(u)| \le M \text{ for all } u \in R,\ M > 0 \text{ a constant.} \tag{2.2}$$

We shall also assume that

$$g(u) = \begin{cases} \gamma_1 u + b_1(u) & \text{if } u \le 0 \\ \gamma_2 u + b_2(u) & \text{if } u > 0, \end{cases} \tag{2.3}$$

with real constants γ_1, γ_2, and $b_1(u)$, $b_2(u)$ bounded for all $u \in R$. Notice that we admit the case of $\gamma_2 = \gamma_1$, and in particular we allow bounded $g(u)$, if $\gamma_2 = \gamma_1 = 0$.

If $u(x) \in H^2(U) \cap H_0^1(U)$ is a solution of (2.1), we decompose it as

$$u(x) = \sum_{i=1}^{n} \xi_i \varphi_i + U_\xi(x), \tag{2.4}$$

where $U_\xi(x)$ is orthogonal in $L^2(U)$ to $\varphi_1, \ldots, \varphi_n$. The following lemma gives an estimate of $U_\xi(x)$, uniformly in ξ_1 and $\mu = (\mu_1, \ldots, \mu_n)$.

Lemma 2.2. *Assume $g(u)$ satisfies the conditions* (2.2) *and* (2.3), *and*

$$kM < \lambda_{n+1}. \tag{2.5}$$

Assume that a solution of (2.1) *is written in the form* (2.4). *Assume that the point $(\xi_2, \ldots, \xi_n)$ belongs to a compact set in R^{n-1}. Then there is a constant $c > 0$, so that*

$$\int_U |\nabla U_\xi|^2 \, dx \le c, \quad \text{uniformly in } \xi_1 \in R, \text{ and } \mu \in R^n. \tag{2.6}$$

Proof. Using the anzatz (2.4) in (2.1) we have

$$\Delta U_\xi + kg(\sum_{i=1}^{n} \xi_i \varphi_i + U_\xi) = \sum_{i=1}^{n} (\mu_i + \lambda_i \xi_i)\varphi_i + e(x) \text{ for } x \in U, \tag{2.7}$$
$$U_\xi = 0 \text{ on } \partial U.$$

We multiply (2.7) by U_ξ, and integrate

$$\int_U |\nabla U_\xi|^2 \, dx - k \int_U g(\sum_{i=1}^{n} \xi_i \varphi_i + U_\xi) U_\xi \, dx = -\int_U e(x) U_\xi \, dx. \tag{2.8}$$

Since $\varphi_1 > 0$, we can find $N > 0$, so that $\sum_{i=1}^{n} \xi_i \varphi_i > 0$ for $\xi_1 > N$, and $\sum_{i=1}^{n} \xi_i \varphi_i < 0$ for $\xi_1 < -N$. Assume that $\xi_1 > N$ (the case $\xi_1 < -N$

is similar, and the case $\xi_1 \in [-N, N]$ is easier, since then $g(\sum_{i=1}^n \xi_i \varphi_i)$ is bounded). We write the second term on the left in (2.8) as

$$-k \int_U \Big[g(\sum_{i=1}^n \xi_i\varphi_i + U_\xi) - g(\sum_{i=1}^n \xi_i\varphi_i)\Big] U_\xi \, dx \tag{2.9}$$
$$- k \int_U \gamma_2 \Big(\sum_{i=1}^n \xi_i\varphi_i\Big) U_\xi \, dx - k \int_U b_2\Big(\sum_{i=1}^n \xi_i\varphi_i\Big) U_\xi \, dx.$$

(With its argument positive, $g(\sum_{i=1}^n \xi_i\varphi_i) = \gamma_2 \sum_{i=1}^n \xi_i\varphi_i + b_2(\sum_{i=1}^n \xi_i\varphi_i)$.) The second integral in (2.9) vanishes. Using the condition (2.2), we estimate the first integral in (2.9) from below by

$$-kM \int_U U_\xi^2 \, dx.$$

Using Lemma 2.1, we then conclude from (2.8) that

$$(\lambda_{n+1} - kM) \int_U U_\xi^2 \, dx \le \int_U |\nabla U_\xi|^2 \, dx - kM \int_U U_\xi^2 \, dx \tag{2.10}$$
$$\le - \int_U e(x) U_\xi \, dx + k \int_U b_2(u) U_\xi \, dx.$$

This gives us an estimate of $\int_U U_\xi^2 \, dx$. Returning to (2.10), we conclude the lemma. □

Remarks. 1) Assume for $\xi \in V \subset R^n$ we have $\sum_{i=1}^n \xi_i\varphi_i > 0$ for all $x \in U$. Then the estimate (2.6) is uniform with respect to $\xi \in V$ and $\mu \in R^n$. (The conclusion is the same if $\sum_{i=1}^n \xi_i\varphi_i < 0$ for all $\xi \in V$ and $x \in U$.)

2) If $\gamma_1 = \gamma_2 = \gamma$ in (2.3), i.e., when $g(u) = \gamma u + b(u)$, with $b(u)$ bounded on R, the estimate (2.6) is uniform with respect to $\xi \in R^n$ and $\mu \in R^n$.

In the following linear problem the function $a(x)$ is given, while $\mu_1, \dots, \mu_n$ and $w(x)$ are unknown.

Lemma 2.3. *Consider the problem*

$$\Delta w + a(x) w = \mu_1 \varphi_1 + \ldots + \mu_n \varphi_n \;\; \textit{for } x \in U, \tag{2.11}$$
$$w = 0 \;\; \textit{on } \partial U,$$
$$\int_U w\varphi_1 \, dx = \ldots = \int_U w\varphi_n \, dx = 0.$$

Assume that

$$a(x) < \lambda_{n+1} \;\; \textit{for all } x \in U. \tag{2.12}$$

Then the only solution of (2.11) *is* $\mu_1 = \ldots = \mu_n = 0$ *and* $w(x) \equiv 0$.

Proof. Multiply the equation in (2.11) by $w(x)$, a solution of the problem (2.11), and integrate. Using Lemma 2.1 and the assumption (2.12), we have

$$\lambda_{n+1} \int_U w^2 \, dx \le \int_U |\nabla w|^2 \, dx = \int_U a(x) w^2 \, dx < \lambda_{n+1} \int_U w^2 \, dx.$$

It follows that $w(x) \equiv 0$, and thus

$$0 = \mu_1 \varphi_1 + \ldots + \mu_n \varphi_n \ \text{ for } x \in U,$$

which implies that $\mu_1 = \ldots = \mu_n = 0$. $\square$

Corollary 1. *If one considers the problem* (2.11) *with* $\mu_1 = \ldots = \mu_n = 0$, *then* $w(x) \equiv 0$ *is the only solution of that problem.*

3. A direct approach to an inverse problem

For the problem (2.1) let us pose an inverse problem: keeping $e(x)$ fixed, find $\mu = (\mu_1, \ldots, \mu_n)$ so that the problem (2.1) has a solution of any prescribed n- signature $\xi = (\xi_1, \ldots, \xi_n)$.

Theorem 3.1. *For the problem* (2.1) *assume that conditions* (2.2), (2.3), *and* (2.5) *hold. Then, given any* $\xi = (\xi_1, \ldots, \xi_n)$, *one can find a unique* $\mu = (\mu_1, \ldots, \mu_n)$ *for which the problem* (2.1) *has a solution of* n*- signature* ξ. *This solution is unique.*

Proof. When $k = 0$, the unique solution of (2.1) can be found in the form $u = \sum_{k=1}^{\infty} \xi_k \varphi_k$. Choosing $\mu_i = -\lambda_i \xi_i$, $i = 1, \ldots, n$, will provide us with a unique solution of the linear problem (2.1) (here $k = 0$) of signature ξ. We now show that solutions of fixed signature ξ can be continued in k.

We begin by assuming that $\xi = \mathbf{0} = (0, \ldots, 0)$. Define $H^2_{\mathbf{0}}$ to be the subspace of $H^2(U) \cap H^1_0(U)$ with zero n-signature:

$$H^2_{\mathbf{0}} = \left\{ u \in H^2(U) \cap H^1_0(U) : \int_U u \varphi_i \, dx = 0, \ i = 1, \ldots, n \right\}.$$

We recast the problem (2.1) in the operator form as

$$F(u, \mu, k) \equiv \Delta u + k g(u) - \mu_1 \varphi_1 - \ldots - \mu_n \varphi_n = e(x),$$

where $F(u, \mu, k) : H^2_{\mathbf{0}} \times R^n \times R \to L^2(U)$. We will show that the implicit function theorem applies, allowing us to continue (u, μ) as a function of k. Compute the Frechet derivative

$$F_{(u,\mu)}(u, \mu, k)(w, \mu^*) = \Delta w + k g'(u) w - \mu_1^* \varphi_1 - \ldots - \mu_n^* \varphi_n.$$

We need to show that the map $F_{(u,\mu)}(u, \mu, k)(w, \mu^*)$ is both injective and surjective. Observe that $F_{(u,\mu)}(u, \mu, k)(w, \mu^*) : H^2_{\mathbf{0}} \times R^n \to L^2(U)$.

The equation $F_{(u,\mu)}(u,\mu,k)(w,\mu^*) = 0$ is equivalent to

$$\Delta w + kg'(u)w = \mu_1^*\varphi_1 + \ldots + \mu_n^*\varphi_n \;\text{ for } x \in U,$$
$$w = 0 \;\text{ on } \partial U,$$
$$\int_U w\varphi_1\,dx = \ldots = \int_U w\varphi_n\,dx = 0.$$

By Lemma 2.3, $w(x) \equiv 0$, and $\mu_1^* = \ldots = \mu_n^* = 0$, proving the injectivity.

To show that the linearized map is also surjective, we need to prove that for any $e^*(x) \in L^2(U)$ the problem

$$\Delta w + kg'(u)w - \mu_1^*\varphi_1 - \ldots - \mu_n^*\varphi_n = e^*(x) \;\text{ for } x \in U, \tag{3.1}$$
$$w = 0 \;\text{ on } \partial U,$$
$$\int_U w\varphi_1\,dx = \ldots = \int_U w\varphi_n\,dx = 0$$

has a solution $(w,\mu_1^*,\ldots,\mu_n^*)$. Consider the operator $L : H^2(U) \cap H_0^1(U) \to L^2(U)$ defined by $L[w] \equiv \Delta w + kg'(u)w$ (i.e. $w = 0$ on ∂U). We distinguish between two cases.

Case 1. The operator L is invertible. Then we can write the solution of the equation in (3.1), subject to the boundary condition in the second line of (3.1) as

$$w(x) = \mu_1^* L^{-1}(\varphi_1(x)) + \ldots + \mu_n^* L^{-1}(\varphi_n(x)) + L^{-1}(e^*(x)).$$

We need to choose $\mu_1^*,\ldots,\mu_n^*$ so that the n conditions in the third line of (3.1) are satisfied; i.e., we have (denoting by $(\cdot,\cdot)$ the inner product in $L^2(U)$)

$$\mu_1^*\left(L^{-1}(\varphi_1(x)),\varphi_1\right) + \ldots + \mu_n^*\left(L^{-1}(\varphi_n(x)),\varphi_1\right) = -\left(L^{-1}(e^*(x)),\varphi_1\right)$$
$$\cdots\;\cdots$$
$$\mu_1^*\left(L^{-1}(\varphi_1(x)),\varphi_n\right) + \ldots + \mu_n^*\left(L^{-1}(\varphi_n(x)),\varphi_n\right) = -\left(L^{-1}(e^*(x)),\varphi_n\right).$$

This is a linear system with unknowns $\mu_1^*,\ldots,\mu_n^*$. If this system was not solvable, its columns would have to be linearly dependent; i.e., we could find constants $c_1,\ldots,c_n$, not all zero, so that

$$\left(L^{-1}\left(c_1\varphi_1(x) + \ldots + c_n\varphi_n(x)\right), \varphi_i(x)\right) = 0, \quad i = 1,\ldots,n.$$

By Lemma 2.3 it follows that $L^{-1}\left(c_1\varphi_1(x) + \ldots + c_n\varphi_n(x)\right) = 0$, but then $0 = c_1\varphi_1(x) + \ldots + c_n\varphi_n(x)$, contradicting the linear independence of the φ_i's.

Case 2. The operator L is not invertible. Since L is a Fredholm operator of index zero , it has a non-trivial kernel. Assume first that the kernel is one dimensional, spanned by some non-zero function $\bar{w}(x)$. In that case $L^{-1}\left(f(x)\right)$ exists if and only if $f(x)$ is orthogonal to the kernel; i.e., $(f(x), \bar{w}(x)) = 0$.

By Corollary 1 to Lemma 2.3 $\bar{w}(x)$ cannot be orthogonal to all of the first n eigenfunctions. Assume for definiteness that $(\bar{w}(x), \varphi_1(x)) \neq 0$. We can then choose the constants $\gamma_1, \dots, \gamma_{n-1}$ such that

$$\begin{aligned} &(\gamma_1\varphi_1(x) + \varphi_2(x)\,, \bar{w}(x)) = 0 \\ &\dots\ \dots \\ &(\gamma_{n-1}\varphi_1(x) + \varphi_n(x)\,, \bar{w}(x)) = 0. \end{aligned} \tag{3.2}$$

We also choose γ_n such that

$$(\gamma_n\varphi_1(x) + e^*(x), \bar{w}(x)) = 0. \tag{3.3}$$

We rewrite the equation in (3.1) as

$$\begin{aligned} \Delta w + kg'(u)w = t_1\left(\gamma_1\varphi_1(x) + \varphi_2(x)\right) + \dots \\ + t_{n-1}\left(\gamma_{n-1}\varphi_1(x) + \varphi_n(x)\right) + \gamma_n\varphi_1(x) + e^*(x). \end{aligned} \tag{3.4}$$

We shall choose the constants $t_1, \dots, t_{n-1}$ to obtain a solution of the problem (3.1). Choosing t_i's is of course equivalent to choosing $\mu_1^*, \dots, \mu_n^*$ in (3.1). In view of (3.2) and (3.3) the following functions are well defined: $w_1 \equiv L^{-1}\left(\gamma_1\varphi_1(x) + \varphi_2(x)\right)$, ..., $w_{n-1} \equiv L^{-1}\left(\gamma_{n-1}\varphi_1(x) + \varphi_n(x)\right)$, and $f \equiv L^{-1}\left(\gamma_n\varphi_1(x) + e^*(x)\right)$. In terms of these functions we write the solution of (3.4) subject to the boundary condition (the second line in (3.1)) as

$$w = t_1w_1 + \dots t_{n-1}w_{n-1} + f + s\bar{w},$$

where s is an arbitrary constant. We now choose the constants $t_1, \dots, t_{n-1}$, s to satisfy n orthogonality conditions in the third line of (3.1). For that we need to solve the linear system

$$\begin{aligned} &t_1\left(w_1, \varphi_1\right) + \dots + t_{n-1}\left(w_{n-1}, \varphi_1\right) + s\left(\bar{w}, \varphi_1\right) = -\left(f, \varphi_1\right) \\ &\dots\ \dots \\ &t_1\left(w_1, \varphi_n\right) + \dots + t_{n-1}\left(w_{n-1}, \varphi_n\right) + s\left(\bar{w}, \varphi_n\right) = -\left(f, \varphi_n\right). \end{aligned}$$

This system is uniquely solvable, unless columns of its matrix are linearly dependent. In that case one can find constants $c_1, \dots, c_n$, not all zero, so that $W \equiv c_1w_1 + \dots + c_{n-1}w_{n-1} + c_n\bar{w}$ is orthogonal to $\varphi_1, \dots, \varphi_n$. But

$$W = L^{-1}\left[c_1\left(\gamma_1\varphi_1(x) + \varphi_2(x)\right) + \dots + c_{n-1}\left(\gamma_{n-1}\varphi_1(x) + \varphi_n(x)\right)\right].$$

We see that $L[W]$ is a linear combination of $\varphi_1, \ldots, \varphi_n$:

$$L[W] = (c_1\gamma_1 + \ldots + c_{n-1}\gamma_{n-1})\,\varphi_1 + c_1\varphi_2 + \ldots c_{n-1}\varphi_n\,.$$

It follows that W satisfies all three lines in (2.11). By Lemma 2.3, $W = 0$, and $L[W] = 0$. The second of these conclusions implies that $c_1 = \ldots = c_{n-1} = 0$. Since $W = 0$, either $c_n = 0$ or $\bar{w} \equiv 0$, and both cases are impossible.

Let us now consider the general case, when the null space of L is multidimensional, spanned by $\bar{w}_1, \ldots, \bar{w}_k$. We have $k \le n$, in view of the condition (2.5). (Zero is an eigenvalue of $L[w] = \lambda w$. By (2.5) it is either the n-th eigenvalue, or lower. Since we repeat each eigenvalue according to its multiplicity, the multiplicity of $\lambda = 0$ is $\le n$.) Assume first that $k < n$. Let $\hat{w}_1, \ldots, \hat{w}_k$ be projections of respectively $\bar{w}_1, \ldots, \bar{w}_k$ onto $X_n \equiv \mathit{span}\,\{\varphi_1, \ldots, \varphi_n\}$. By Corollary 1 to Lemma 2.3 the $\bar{w}_i$'s are all non-zero. We can find mutually orthogonal functions $\psi_1, \ldots, \psi_{n-k} \in X_n$, which are orthogonal to $\hat{w}_1, \ldots, \hat{w}_k$, and hence they are also orthogonal to $\bar{w}_1, \ldots, \bar{w}_k$. Without loss of generality we may assume that $e^* \in X_n^{\perp}$ (otherwise one can absorb into μ_i^* the projection of e^* onto φ_i). Write the equation in (3.1) as

$$L[w] = t_1\psi_1 + \ldots + t_{n-k}\psi_{n-k} + e^*. \tag{3.5}$$

We shall choose the constants $t_1, \ldots, t_{n-k}$ to obtain a solution of (3.1) (choosing t_i's is equivalent to choosing μ_i^*'s). The solution of (3.5) is

$$w = t_1 w_1 + \ldots + t_{n-k} w_{n-k} + f + s_1 \bar{w}_1 + \ldots + s_k \bar{w}_k,$$

where we denote $w_i = L^{-1}[\psi_i]$ and $f = L^{-1}[e^*]$. We need to choose n constants $t_1, \ldots, t_{n-k}$ and $s_1, \ldots, s_k$ so that $w \perp X_n$; i.e., we need to solve the linear system

$$t_1\,(w_1, \varphi_1) + \ldots + t_{n-k}\,(w_{n-k}, \varphi_1) + s_1\,(\bar{w}_1, \varphi_1) + \ldots + s_k\,(\bar{w}_k, \varphi_1) = -\,(f, \varphi_1)$$
$$\ldots\;\ldots$$
$$t_1\,(w_1, \varphi_n) + \ldots + t_{n-k}\,(w_{n-k}, \varphi_n) + s_1\,(\bar{w}_1, \varphi_n) + \ldots + s_k\,(\bar{w}_k, \varphi_n) = -\,(f, \varphi_n)\,.$$

The system is uniquely solvable, unless its columns are linearly dependent. If that was the case, we could find constants $c_1, \ldots, c_{n-k}, d_1, \ldots, d_k$, not all zero, so that

$$W \equiv c_1 w_1 + \ldots + c_{n-k} w_{n-k} + d_1 \bar{w}_1 + \ldots + d_k \bar{w}_k \perp X_n.$$

Observe that

$$L[W] = c_1\psi_1 + \ldots + c_{n-k}\psi_{n-k} \in X_n. \tag{3.6}$$

Applying Lemma 2.3 to the problem (3.6) we conclude that $W = 0$ and

$$0 = c_1\psi_1 + \ldots + c_{n-k}\psi_{n-k}.$$

Since the ψ_i's are mutually orthogonal, we conclude that c_i's are all zero. Combining that with the fact that $W = 0$, we see that

$$d_1\bar{w}_1 + \ldots + d_k\bar{w}_k = 0,$$

and hence all d_i's are also zero, a contradiction.

If $k = n$ we proceed similarly. In the linear system above, there are no t_i's, while we have $s_1, \ldots, s_n$. Linear independence of columns would imply existence of constants $d_1, \ldots, d_n$, not all zero, so that for $W \equiv d_1\bar{w}_1 + \ldots + d_n\bar{w}_n$ we have $L[W] = 0$, and $W \perp X_n$. By Corollary 1 to Lemma 2.3, we have $W = 0$, contradicting linear independence of the $\bar{w}_i$'s.

We now consider the case of general ξ, and reduce it to the case $\xi = 0$, by setting $v(x) = u(x) - \sum_{i=1}^n \xi_i\varphi_i$. Then $v(x)$ satisfies

$$\begin{aligned} &\Delta v + kg(v + \sum_{i=1}^n \xi_i\varphi_i) = \sum_{i=1}^n (\mu_i + \lambda_i\xi_i)\varphi_i + e(x) \ \text{ for } x \in U, \\ &u = 0 \ \text{ on } \partial U. \end{aligned} \tag{3.7}$$

Even though this problem is slightly different from (2.1), it is clear that we can repeat the above argument, and obtain a curve of solutions of (3.7) of zero signature. Then $u = u(x,k) \equiv u(k)$ is a solution curve of fixed signature ξ for the problem (2.1).

Hence, the implicit function theorem applies at any solution of (2.1); i.e., locally we have a curve of solutions $u = u(k)$, $\mu_i = \mu_i(k)$, $i = 1, \ldots, n$. To show that this curve can be continued for all k, satisfying our condition (2.5), we only need to show that this curve $(u(k), \mu(k))$ cannot go to infinity at some k; i.e., we need an a priori estimate. Since the n-signature of the solution is fixed, we only need to estimate U_ξ. We claim that there is a constant $c > 0$, so that

$$||U_\xi||_{H^2(U)} \le c, \ \text{ uniformly in } \mu \in R^n. \tag{3.8}$$

Indeed, multiply (2.7) by ΔU_ξ and integrate. Since $\Delta U_\xi \perp X_n$, we have

$$\begin{aligned} &\int_U (\Delta U_\xi)^2\,dx + k\int_U \Big[g(\sum_{i=1}^n \xi_i\varphi_i + U_\xi) - g(0)\Big]\Delta U_\xi\,dx \\ &= \int_U (e(x) - kg(0))\,\Delta U_\xi\,dx. \end{aligned} \tag{3.9}$$

The second term we integrate by parts, obtaining

$$-\int_U g'(\sum_{i=1}^n \xi_i\varphi_i + U_\xi)\Big(\sum_{i=1}^n \xi_i\nabla\varphi_i + \nabla U_\xi\Big)\cdot\nabla U_\xi\,dx.$$

(There is no boundary term, since the quantity in the square bracket vanishes on ∂U.) Since $g'(u)$ is uniformly bounded, we get by Lemma 2.2 a bound for this integral. Returning to (3.9), we have a bound on $\int_U (\Delta U_\xi)^2\, dx$, and by elliptic regularity we conclude (3.8). (The estimate (3.8) is not uniform in ξ, but we keep the signature ξ fixed.) Multiplying (2.1) by φ_i and integrating over U, we conclude the existence of a uniform bound on $\mu_i(k)$. It follows that we can continue the solutions of the problem (2.1), of fixed signature ξ, for all k satisfying the condition (2.5). At each k we have a solution $(u(k), \mu(k))$ of (2.1).

Finally, if the problem (2.1) had a different solution $(\bar{u}(k), \bar{\mu}(k))$ with the same signature ξ, we would continue it back in k, obtaining at $k = 0$ a different (since solution curves do not intersect) solution of the linear problem of signature ξ, which is impossible. □

4. Continuation in ξ for k fixed

Theorem 3.1 implies that the value of $\xi = (\xi_1, \ldots, \xi_n)$ uniquely identifies the solution pair $(\mu, u(x))$, where $\mu = (\mu_1, \ldots, \mu_n)$. Hence the solution set of (2.1) can be faithfully described by the map $\xi \in R^n \to \mu \in R^n$, which we call the *solution manifold*. (If $n = 1$ we have a *solution curve* $\mu = \mu(\xi)$, which faithfully depicts the solution set.) We show next that the solution manifold is connected.

Theorem 4.1. *In the conditions of Theorem* 3.1 *any solution of* n*- signature* ξ *can be locally continued in* ξ. *This continuation can in fact be performed globally. In particular, we can continue solutions of any signature* $\bar{\xi}$ *to a solution of arbitrary signature* $\hat{\xi}$ *by following any continuous curve joining* $\bar{\xi}$ *and* $\hat{\xi}$.

Proof. We show that any solution of (2.1) can be continued in ξ, by using the implicit function theorem. The proof is essentially the same as for continuation in k above. Letting $u(x) = \sum_{i=1}^n \xi_i \varphi_i + v(x)$, we recast the problem (2.1) in the operator form $F(v, \mu, \xi) = e(x)$, where $F : H^2_{\mathbf{0}} \times R^n \times R^n \to L^2$ is defined by

$$F(v, \mu, \xi) = \Delta v + kg(\sum_{i=1}^{n} \xi_i \varphi_i + v(x)) - (\mu_1 + \lambda_1 \xi_1)\varphi_1 - \ldots - (\mu_n + \lambda_n \xi_n)\varphi_n.$$

(Observe that the first n harmonics of v are zero.) The Frechet derivative this time is

$$F_{(v,\mu)}(v, \mu, \xi)(w, \mu^*) = \Delta w + kg'(\sum_{i=1}^{n} \xi_i \varphi_i + v(x))w - \mu_1^* \varphi_1 - \ldots - \mu_n^* \varphi_n,$$

with $w \in H^2_{\mathbf{0}}$, and $\mu^* \in R^n$. According to Lemma 2.3 the only solution of the linearized problem

$$\begin{aligned}
&F_{(v,\mu)}(v,\mu,\xi)(w,\mu^*) = 0 \ \text{ for } x \in U,\\
&w = 0 \ \text{ on } \partial U,\\
&w \in H^2_{\mathbf{0}}
\end{aligned}$$

is $(w,\mu^*) = (0,\mathbf{0})$. As before, we verify the surjectivity of the map $F_{(v,\mu)}(v,\mu,\xi)(w,\mu^*)$. Hence, the implicit function theorem applies, giving us locally a curve of solutions $u = u(x,\xi)$ and $\mu = \mu(\xi)$. As before, we see that solutions on this curve remain bounded, and hence we can continue the curve for all $\xi \in R^n$. (The bound (3.8) depended on ξ. However, once we fix two points $\bar{\xi}$ and $\hat{\xi}$, and the path joining them, the bound (3.8) is uniform.) $\square$

5. Solution manifold in the case $n = 1$

The results of this section are largely known. However, we derive them in a unified fashion, and we provide more detailed information on the solution curves, which opens a way for numerical computations. Moreover, we obtain extensions of most of these results.

We begin the study of the solution manifold in the case $n = 1$, i.e., when only the first harmonic (of both the forcing term and of the solution) is considered separately. The solution manifold in this case is a curve, and we study its properties in this section. We can treat asymptotically linear problems of the form (here λ is a parameter)

$$\begin{aligned}
&\Delta u + \lambda u + b(u) = \mu_1\varphi_1 + e(x) \ \text{ for } x \in U, \qquad (5.1)\\
&u = 0 \ \text{ on } \partial U,
\end{aligned}$$

where $e(x)$ is orthogonal to φ_1, and $b(u) \in C^1(R)$ is a bounded function, satisfying

$$\lambda + b'(u) < \lambda_2 \quad \text{for all } u \in R. \qquad (5.2)$$

This condition is assumed throughout the present section. By Theorem 3.1, for any $\xi_1 \in R$ one can find a unique solution pair $\mu_1 = \mu_1(\xi_1)$ and $u = u(x,\xi_1) = \xi_1\varphi_1 + U_{\xi_1}$ (k is now fixed). When one varies ξ_1, all these solutions link up to form a unique solution curve, which exhausts the solution set of (5.1). Indeed, by Theorem 4.1 we can continue solutions for $-\infty < \xi_1 < \infty$, giving us a solution curve, and no solutions off this curve are possible, since the value of ξ_1 uniquely identifies the solution, and all values of ξ_1 have been accounted for.

We now study the range of the function $\mu_1 = \mu_1(\xi_1)$. If this range is $(-\infty, \infty)$, it follows that the problem (5.1) is solvable for all right-hand sides $f(x) \equiv \mu_1\varphi_1 + e(x) \in L^2$.

Proposition 1. *Assume that $\lambda \neq \lambda_1$. Then the range of the function $\mu_1 = \mu_1(\xi_1)$ is $(-\infty, \infty)$.*

Proof. Multiplying (5.1) by φ_1, and integrating gives

$$(\lambda - \lambda_1)\xi_1 + \int_U b(u)\varphi_1 \, dx = \mu_1. \tag{5.3}$$

Since $b(u)$ is bounded, it follows that μ_1 goes to infinity when ξ_1 does. □

Observe that, if $\lambda > \lambda_1$, μ_1 goes to infinity in the same direction as ξ_1, while for $\lambda < \lambda_1$ the direction is opposite. Clearly, something must happen at $\lambda = \lambda_1$ to allow this drastic change. It turns out that μ_1 is bounded at $\lambda = \lambda_1$. The following observation follows from (5.3).

Proposition 2. *If $\lambda = \lambda_1$, then a solution of (5.1) may exist only if*

$$\inf_R b(u) \int_U \varphi_1 \, dx \leq \mu_1 \leq \sup_R b(u) \int_U \varphi_1 \, dx.$$

We can get a more detailed result if we assume that $b(u)$ has limits at infinity $b(-\infty)$ and $b(\infty)$. This sort of result has originated from the celebrated paper of E.M. Landesman and A.C. Lazer [12].

Theorem 5.1. *If the limits $b(\pm\infty)$ exist, then*

$$\lim_{\xi_1 \to \infty} \mu_1(\xi_1) = b(\infty) \int_U \varphi_1 \, dx, \quad \textit{and}$$

$$\lim_{\xi_1 \to -\infty} \mu_1(\xi_1) = b(-\infty) \int_U \varphi_1 \, dx.$$

The problem (5.1) *is solvable, provided*

$$\min\left(b(-\infty), b(\infty)\right) \int_U \varphi_1 \, dx < \mu_1 < \max\left(b(-\infty), b(\infty)\right) \int_U \varphi_1 \, dx. \tag{5.4}$$

Proof. We prove that the limits exist. Then (5.4) will follow, since the curve $\mu_1 = \mu_1(\xi_1)$ is continuous. Assume that $\xi_1 \to \infty$. Recall that we decompose $u = \xi_1\varphi_1 + U_\xi$. From (5.3) (at $\lambda = \lambda_1$) for any $\delta > 0$ and small

$$|\mu_1 - b(\infty) \int_U \varphi_1 \, dx| \leq \int_U |b(\xi_1\varphi_1 + U_\xi)\varphi_1 - b(\infty)\varphi_1| \, dx = \tag{5.5}$$

$$\int_{\varphi_1 \leq \delta} |b(\xi_1\varphi_1 + U_\xi)\varphi_1 - b(\infty)\varphi_1| \, dx + \int_{\varphi_1 \geq \delta} |b(\xi_1\varphi_1 + U_\xi)\varphi_1 - b(\infty)\varphi_1| \, dx.$$

Since $b(u)$ is bounded, the first integral on the right can be made less than $\epsilon/3$ by choosing δ small. The second integral we decompose over two complementary sets, keeping the same integrand (which we do not show)

$$\int_{|U_\xi|<\frac{\xi_1\varphi_1}{2}} + \int_{|U_\xi|>\frac{\xi_1\varphi_1}{2}} . \tag{5.6}$$

In the second integral in (5.6) we integrate over the set where $|U_\xi| > \frac{\xi_1\delta}{2}$. Since U_ξ is bounded in L^2 uniformly in ξ_1 by Lemma 2.2, the measure of this set tends to zero, for ξ_1 large, which allows us to make the second integral in (5.6) less than $\epsilon/3$. The first integral in (5.6) can also be made less than $\epsilon/3$, since over its region of integration $\xi_1\varphi_1 + U_\xi > \frac{\xi_1\delta}{2}$, which is large for ξ_1 large. □

Example. Consider the problem (5.1). Assume that (5.2) holds, and in addition $b(u)$ is bounded, $b(\pm\infty) = 0$, and $b(u) > 0$ for all $u \in R$. Then there exists $\alpha > 0$ so that the problem (5.1) is solvable if and only if $\mu_1 \in (0, \alpha]$. Moreover, the problem has at least two solutions for any $\mu_1 \in (0, \alpha)$. Indeed, we see that $\mu_1(\xi_1) > 0$ for all ξ_1, and, arguing as above, $\mu_1(\xi_1) \to 0$ as $\xi_1 \to \pm\infty$. Hence the range of $\mu_1(\xi_1)$ is $(0, \alpha]$, and it is covered at least twice. This generalizes Problem 13 on page 162 in A. Ambrosetti and G. Prodi [3] (the $e(x)$ term was not included there). Moreover, one can easily extend this result by dropping the $b(u) > 0$ condition. (Then there exists $\beta \le 0 < \alpha$, so that the problem (5.1) is solvable if and only if $\mu_1 \in [\beta, \alpha]$. If $\mu_1 \in (\beta, \alpha)$ the problem has at least two solutions, except for the possibility of exactly one solution at $\mu_1 = 0$.)

For monotone $b(u)$ we have a uniqueness result.

Theorem 5.2. *In addition to the assumptions of Theorem* 5.1 *assume that* $b'(u) > 0$ (< 0) *for all* $u \in R$. *Then* $\mu_1'(\xi_1) > 0$ (< 0) *for all* $\xi_1 \in R$, *which implies that for any* μ_1 *satisfying the necessary condition* (5.4) *the problem* (5.1) *has a unique solution.*

Proof. Assume for definiteness that $b'(u) > 0$ for all $u \in R$. Then, by Theorem 5.1, $\mu_1(\infty) > \mu_1(-\infty)$, and hence $\mu_1'(\xi_1) > 0$ at least for some values of ξ_1. Assume that $\mu_1'(\xi_1^0) = 0$ at some ξ_1^0. Differentiate (5.1) in ξ_1, set $\xi_1 = \xi_1^0$, and denote $w = u_{\xi_1}|_{\xi_1=\xi_1^0}$, obtaining

$$\begin{aligned} &\Delta w + \big(\lambda_1 + b'(u)\big)\, w = 0 \ \text{ for } x \in U, \\ &w = 0 \ \text{ on } \partial U. \end{aligned}$$

Clearly, w is not zero, since it has a non-zero projection on φ_1. On the other hand, $w \equiv 0$, since by the assumption (5.2) $\lambda_1 < \lambda_1 + b'(u) < \lambda_2$. □

Another way to "do business" at resonance, without imposing conditions of Landesman-Lazer type, is due to D.G. de Figueiredo and W.-M. Ni [8]. Consider again the problem (5.1) at $\lambda = \lambda_1$, with $b(u)$ bounded, and satisfying (5.2). Assume also that $b(u)u > 0$ for all $u \in R$. From (5.3)

$$\mu_1 = \int_U b(u)\varphi_1 \, dx.$$

Arguing as in Theorem 5.1, we see that $\int_U b(u)\varphi_1 \, dx > 0$ (< 0) if $\xi_1 > 0$ (< 0) and large. It follows that the range of $\mu_1(\xi_1)$ includes an interval around zero. In particular the problem (5.1) is solvable at $\mu = 0$, which corresponds to the result of [8] (we obtain an extension of that result, allowing for μ to be non-zero, but not a generalization, since in [8] they had milder assumptions on $b(u)$). We can also extend the positivity condition, by assuming that $b(u) > \delta$ for u positive and large, and $b(u) < -\delta$ for u negative and large, for some $\delta > 0$.

The next case to consider is that of jumping nonlinearities. Problems of this type have been extensively studied, with S. Fucik [9] being one of the earliest contributions. We consider the problem

$$\begin{aligned} &\Delta u + g(u) = \mu_1\varphi_1 + e(x) \quad \text{for } x \in U, \\ &u = 0 \quad \text{on } \partial U, \end{aligned} \tag{5.7}$$

where

$$g(u) = \begin{cases} \gamma_1 u + b_1(u) & \text{if } u \le 0 \\ \gamma_2 u + b_2(u) & \text{if } u > 0, \end{cases} \tag{5.8}$$

with real constants γ_1, γ_2, and $b_1(u)$, $b_2(u)$ bounded for all $u \in R$. To apply Theorem 4.1, we need $g'(u) < \lambda_2$; i.e., we assume that $\gamma_1 + b_1'(u), \gamma_2 + b_2'(u) < \lambda_2$ for all $u \in R$. There are several cases to consider.

Case (i) $\gamma_1, \gamma_2 < \lambda_1$. From (5.7)

$$\mu_1 = -\lambda_1\xi_1 + \int_U g(u)\varphi_1 \, dx. \tag{5.9}$$

Arguing similarly to Theorem 5.1, we see that, for large ξ_1, μ is asymptotic to $(\gamma_2 - \lambda_1)\xi_1$, and hence $\lim_{\xi_1 \to \infty} \mu_1(\xi_1) = -\infty$. Similarly, as $\xi_1 \to -\infty$, $\mu_1(\xi_1)$ is asymptotic to $(\gamma_1 - \lambda_1)\xi_1$, and hence $\lim_{\xi_1 \to -\infty} \mu_1(\xi_1) = \infty$. It follows that the range of $\mu_1(\xi_1)$ is $(-\infty, \infty)$, and hence the problem (5.7) is solvable for all μ_1.

Case (ii) $\lambda_1 < \gamma_1, \gamma_2 < \lambda_2$. We have similar conclusions, although now $\mu_1(\xi_1) \to \pm\infty$ as $\xi_1 \to \pm\infty$.

Case (iii) $\gamma_1 < \lambda_1 < \gamma_2 < \lambda_2$. It turns out that the range of $\mu_1(\xi_1)$ is bounded on one side. Arguing as above, we see from (5.9) that $\mu_1(\xi_1) \to \infty$

as $\xi_1 \to \pm\infty$. Hence the range of $\mu_1(\xi_1)$ is $[\mu_0, \infty)$, for some μ_0. It follows that for $\mu \geq \mu_0$ the problem (5.7) has a solution, and no solution exists when $\mu < \mu_0$.

Case (iv) $\gamma_2 < \lambda_1 < \gamma_1 < \lambda_2$. This case is similar to (iii). This time $\mu_1(\xi_1)$ is bounded from above.

Referring to (5.8), define $b(u) = \begin{cases} b_1(u) & \text{if } u \leq 0 \\ b_2(u) & \text{if } u > 0. \end{cases}$ Let us now assume that case (iii) holds, while $b(u) \in C^2(R)$, and satisfies

$$b''(u) > 0 \text{ (and hence } g''(u) > 0) \text{ for almost all } u \in R. \tag{5.10}$$

Then $\mu_1(\xi_1)$ is bounded from below, and in fact we have the following precise result, which is roughly equivalent to the one in the classical papers of A. Ambrosetti and G. Prodi [2] and M. S. Berger and E. Podolak [4].

Theorem 5.3. *The function $\mu_1(\xi_1)$ has a point of global minimum at some ξ_1^0, and $\mu_1(\xi_1)$ is strictly decreasing on $(-\infty, \xi_1^0)$ and strictly increasing on (ξ_1^0, ∞). The problem (5.7) has exactly two, exactly one or no solution, depending on whether $\mu \in (\mu(\xi_1^0), \infty)$, $\mu = \mu(\xi_1^0)$ or $\mu < \mu(\xi_1^0)$ respectively.*

Proof. Differentiate the equation (5.7) with respect to ξ_1 :

$$\Delta u_{\xi_1} + g'(u)u_{\xi_1} = \mu_1'(\xi_1)\varphi_1 \text{ for } x \in U, \ \ u_{\xi_1} = 0 \text{ on } \partial U. \tag{5.11}$$

Assume that $\mu_1'(\xi_0) = 0$ at some ξ_0. Then, denoting $w = u_{\xi_1}$ at $\xi_1 = \xi_0$, we have

$$\Delta w + g'(u)w = 0 \text{ for } x \in U, \ \ w = 0 \text{ on } \partial U. \tag{5.12}$$

This problem is a linearization of (5.7). Hence, if the problem (5.12) has only the trivial solution, we can continue the solution $u = u(x, \xi_0)$ in μ, obtaining solutions on an interval around $\mu(\xi_0)$; i.e., the function $\mu(\xi_1)$ does not have an extremum at ξ_0.

Assume next that the problem does have a non-trivial solution. Since $g'(u) < \lambda_2$, w is a principal eigenfunction, and so we may assume that $w > 0$ on U. Differentiate (5.11) once more in ξ_1, and let $\xi_1 = \xi_0$ (recall that $\mu_1'(\xi_0) = 0$):

$$\Delta u_{\xi_1\xi_1} + g'(u)u_{\xi_1\xi_1} + g''(u)w^2 = \mu_1''(\xi_0)\varphi_1 \text{ for } x \in U, \ \ u_{\xi_1\xi_1} = 0 \text{ on } \partial U. \tag{5.13}$$

Combining the equations (5.12) and (5.13), we have

$$\mu_1''(\xi_0)\int_U w\varphi_1\,dx = \int_U g''(u)w^3\,dx,$$

which implies that $\mu_1''(\xi_0) > 0$, so that any critical point of $\mu_1(\xi)$ is either a point of minimum, or a point of inflection. Hence this function has only one extremum point, the point of global minimum, and the theorem follows. □

Dropping the convexity assumption, we have the following result, similar to the one in H. Amann and P. Hess [1]. What we add is that all solutions lie on a unique solution curve $\mu_1 = \mu_1(\xi_1)$.

Theorem 5.4. *Consider the problem* (5.7), *with jumping nonlinearity as in case* **(iii)***. Then the function $\mu_1(\xi_1)$ has a point of global minimum at some ξ_1^0. The problem* (5.8) *has at least two, at least one or no solution, depending on whether $\mu \in (\mu(\xi_1^0), \infty)$, $\mu = \mu(\xi_1^0)$ or $\mu < \mu(\xi_1^0)$ respectively.*

Proof. The function $\mu(\xi_1)$ is bounded from below, and hence it has a point of global minimum. We claim that $\lim_{\xi_1 \to \pm\infty} \mu(\xi_1) = \infty$, which clearly implies that the range of $\mu(\xi_1)$ extends to infinity, and is covered twice. Assuming the claim to be false, we can find a sequence $\xi_1^k \to \infty$, such that corresponding $\mu_k \equiv \mu(\xi_1^k)$ are uniformly bounded. Consider now the problem (5.7) with $\mu = \mu_k$. According to Lemma 3.2 in A.C. Lazer and P. J. McKenna [13] corresponding solutions of (5.7), which we call u_k, are uniformly bounded in $C^1(\bar{U})$. We have $u_k = \xi_1^k \varphi_1 + U_{\xi_1}^k$, with $U_{\xi_1}^k$ uniformly bounded in $L^2(U)$ by Lemma 2.2. It follows that the sequence u_k is not bounded in L^2, a contradiction. □

And finally, we consider a sample case of "half-resonance." Consider again the problem (5.7), with $g(u)$ satisfying (5.8). Assume also that $\lambda_1 < \gamma_1 < \lambda_2$, $\gamma_2 = \lambda_1$, and $\lim_{u \to -\infty} b_1(u) = 0$, $\lim_{u \to \infty} b_2(u) = 0$. Then as above we conclude that $\mu_1(\xi_1) \to -\infty$ as $\xi_1 \to -\infty$, and $\mu_1(\xi_1) \to 0$ as $\xi_1 \to \infty$. In particular the problem (5.7) is solvable for any $\mu_1 < 0$. This extends to PDE's (and in other ways too) a result of A. Castro [5]. A. Castro's result had an advantage of not requiring an upper bound on γ_1.

6. Solution manifold for $n > 1$

Let us begin with the case $n = 2$, and then generalize. Again, we consider asymptotically linear problems of the form (here λ is a parameter)

$$\begin{aligned} &\Delta u + \lambda u + b(u) = \mu_1 \varphi_1 + \mu_2 \varphi_2 + e(x) \quad \text{for } x \in U, \\ &u = 0 \quad \text{on } \partial U, \end{aligned} \tag{6.1}$$

where $e(x) \in L^2(U)$ is orthogonal to φ_1 and φ_2, and $b(u) \in C^1(R)$ is a bounded function, satisfying

$$\lambda + b'(u) < \lambda_3 \quad \text{for all } u \in R. \tag{6.2}$$

We first consider the non-resonant case, when $\lambda \neq \lambda_i$, $i = 1, 2$, and show that the problem (6.1) is solvable for any pair of $(\mu_1^0, \mu_2^0) \in R^2$. We decompose the solution as $u = \xi_1\varphi_1 + \xi_2\varphi_2 + U_\xi$. According to the Theorem 3.1 for every pair (ξ_1, ξ_2) we can find (μ_1, μ_2, u) solving (6.1). By Theorem 4.1 we can continue the solution between any two points of (ξ_1, ξ_2) plane. Our goal is to find (ξ_1^0, ξ_2^0) corresponding to (μ_1^0, μ_2^0). From (6.1) we obtain

$$(\lambda - \lambda_1)\xi_1 + \int_U b\left(\xi_1\varphi_1 + \xi_2\varphi_2 + U_\xi\right)\varphi_1\,dx = \mu_1; \tag{6.3}$$

$$(\lambda - \lambda_2)\xi_2 + \int_U b\left(\xi_1\varphi_1 + \xi_2\varphi_2 + U_\xi\right)\varphi_2\,dx = \mu_2. \tag{6.4}$$

If $\lambda > \lambda_1$ ($\lambda < \lambda_1$), we see that $\mu_1 \to \pm\infty$ ($\mu_1 \to \mp\infty$) as $\xi_1 \to \pm\infty$, uniformly in ξ_2, since $b(u)$ is bounded. Let us say $\lambda > \lambda_1$, and we begin continuation somewhere on the line $\xi_1 = -N$ of the (ξ_1, ξ_2) plane. We assume $N > 0$ to be large, and hence $\mu_1(\xi_1, \xi_2)$ is large and negative on this line. We see from (6.4) that μ_2 takes on both large positive and large negative values when $|\xi_2|$ is large. Fix $\xi_2 = \xi_2^0$, so that $\mu_2 = \mu_2^0$; i.e., at the point $(\xi_1, \xi_2) = (-N, \xi_2^0)$ we have $(\mu_1, \mu_2) = (\bar{\mu}_1, \mu_2^0)$, with some $\bar{\mu}_1$. We now begin to shift the line $\xi_1 = -N$ to the right; i.e., we consider the lines $\xi_1 = -N + t$, with $t > 0$. We repeat the above procedure, choosing the point $(\xi_1, \xi_2) = (-N + t, \xi_2^0(t))$ at which $(\mu_1, \mu_2) = (\bar{\mu}_1(t), \mu_2^0)$. Observe that from (6.4) ξ_2^0 is continuous in ξ_1, i.e., continuous in t. From (6.3) we see that $\bar{\mu}_1(t)$ is continuous in ξ_1 and ξ_2^0, i.e., continuous in t. For large t, $\bar{\mu}_1(t)$ will transition to large positive values. Hence at some t, i.e., at some (ξ_1^0, ξ_2^0), we obtain $(\mu_1, \mu_2) = (\mu_1^0, \mu_2^0)$.

We consider next the resonance case $\lambda = \lambda_2$. Assume first that the (finite) limits $b(\pm\infty)$ exist. Set

$$A = b(\infty)\int_{U^+}\varphi_2 + b(-\infty)\int_{U^-}\varphi_2, \quad B = b(-\infty)\int_{U^+}\varphi_2 + b(\infty)\int_{U^-}\varphi_2,$$

where $U^+ = \{x \in U : \varphi_2(x) > 0\}$ and $U^- = \{x \in U : \varphi_2(x) < 0\}$. Arguing as in the proof of Theorem 5.1, we see from (6.4) that $\mu_2 \to A$ as $\xi_2 \to \infty$, and $\mu_2 \to B$ as $\xi_2 \to -\infty$. Then the same argument with the lines $\xi_1 = -N + t$ shows that the problem

$$\begin{aligned} &\Delta u + \lambda_2 u + b(u) = \mu_1^0\varphi_1 + \mu_2^0\varphi_2 + e(x) \quad \text{for } x \in U \\ &u = 0 \quad \text{on } \partial U \end{aligned} \tag{6.5}$$

is solvable for any $\mu_1^0 \in R$, and any $\mu_2^0 \in (\min(A, B), \max(A, B))$.

We also have an analog of D.G. de Figueiredo and W.-M. Ni result [8] (see also Theorem 1.10 in A. Ambrosetti and G. Prodi [3]). If $b(u)$ is bounded below by a positive constant for $u > 0$ and large, and bounded above by a negative constant for $u < 0$ and large, the problem (6.5) is solvable for $\mu_2^0 = 0$ and arbitrary μ_1^0. Indeed, μ_2 has to vanish somewhere on each line $\xi_1 = -N + t$, since it takes both positive and negative values on this line. ($\mu_2 = \int_{U^+} b\,(\xi_1\varphi_1 + \xi_2\varphi_2 + U_\xi)\,\varphi_2\,dx + \int_{U^-} b\,(\xi_1\varphi_1 + \xi_2\varphi_2 + U_\xi)\,\varphi_2\,dx$, with the sets U^+ and U^- as defined above. When $\xi_2 \to \infty$, both integrals are positive, and they are both negative when $\xi_2 \to -\infty$.)

We consider next the case of $n = 3$, after which generalization to general n will be transparent. We consider the problem

$$\begin{aligned}&\Delta u + \lambda u + b(u) = \mu_1\varphi_1 + \mu_2\varphi_2 + \mu_3\varphi_3 + e(x) \text{ for } x \in U, \\ &u = 0 \text{ on } \partial U,\end{aligned} \tag{6.6}$$

where $e(x) \in L^2(U)$ is orthogonal to φ_i, $i = 1, 2, 3$, and $b(u) \in C^1(R)$ is a bounded function, satisfying

$$\lambda + b'(u) < \lambda_4 \quad \text{for all } u \in R. \tag{6.7}$$

We consider the non-resonant case, when $\lambda \neq \lambda_i$, $i = 1, 2, 3$, and show that the problem (6.1) is solvable for any triple $(\mu_1^0, \mu_2^0, \mu_3^0) \in R^3$. In the parameter space (ξ_1, ξ_2, ξ_3) consider the plane $\xi_3 = M$. By above, on any such plane we can find a point (ξ_1^0, ξ_2^0) at which $(\mu_1, \mu_2) = (\mu_1^0, \mu_2^0)$. Multiplying (6.6) by φ_3 and integrating, we see that as M varies over the interval $(-\infty, \infty)$ so does μ_3. Hence we can select M where $\mu = \mu_3^0$ at the point where $(\mu_1, \mu_2) = (\mu_1^0, \mu_2^0)$; i.e., we obtain a solution at the desired point $(\mu_1^0, \mu_2^0, \mu_3^0)$.

7. Symmetry breaking

As an application of our results, we can give a simple proof that for $b > \lambda_2$ the problem

$$\begin{aligned}&\Delta u + bu^+ = \varphi_1 \text{ for } x \in B, \\ &u = 0 \text{ on } \partial B\end{aligned} \tag{7.1}$$

has multiple non-radial solutions. Here B is the unit ball centered at the origin in R^2, b a parameter, and $u^+ = \max(u, 0)$. Recently existence of one non-radial solution was proved (among other things) by F. Pacella and P. N. Srikanth [14] for $b > \lambda_2$, and b sufficiently large, for balls in R^n. (Our approach appears to be applicable for balls in R^n too, but we did not pursue that.)

If the ball B is two dimensional, then the eigenvalues of the Laplacian with zero boundary conditions are $(\mu_j^k)^2$, and the corresponding eigenfunctions are $J_k(\mu_j^k r)\cos k\theta$ and $J_k(\mu_j^k r)\sin k\theta$, where $J_k(r)$ is the k-th Bessel function ($k \geq 0$), and μ_j^k is its j-th root ($j \geq 1$). *Mathematica* readily gives $\mu_1^0 \simeq 2.40483$, $\mu_2^0 \simeq 5.52008$ and $\mu_3^0 \simeq 8.65373$, while $\mu_1^1 \simeq 3.83171$, $\mu_2^1 \simeq 7.01559$, $\mu_3^1 \simeq 10.1735$, and $\mu_1^2 \simeq 5.13562$, $\mu_2^2 \simeq 8.41724$, $\mu_3^2 \simeq 11.6198$. (The command BesselJZeros[n,3] gives the first three roots of J_n.) It follows that $\lambda_1 = (\mu_1^0)^2$ and $\varphi_1 = J_0(\mu_1^0 r)$, $\lambda_2 = (\mu_1^1)^2 = \lambda_3$, with $\varphi_2 = J_1(\mu_1^1 r)\sin\theta$ and $\varphi_3 = J_1(\mu_1^1 r)\cos\theta$. Then one has $\lambda_4 = \lambda_5 = (\mu_1^2)^2$, and so on. We have $\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 = \lambda_5 \ldots$.

Theorem 7.1. *For any $\lambda_2 < b < \lambda_4$ the problem* (7.1) *has four non-radial solutions.*

Proof. We shall prove that the problem (7.1) has a solution with non-zero projections on φ_2 and φ_3 of the form

$$u = \xi_1\varphi_1 + \xi_2\varphi_2 + \xi_2\varphi_3 + U_\xi, \tag{7.2}$$

with $\xi_2 \neq 0$. By Theorem 3.1 for any triple (ξ_1, ξ_2, ξ_2) we can find a *unique* triple (μ_1, μ_2, μ_3) and $u(x)$ solving

$$\Delta u + bu^+ = \mu_1\varphi_1 + \mu_2\varphi_2 + \mu_3\varphi_3 \ \text{ for } x \in B, \ \ u = 0 \ \text{ on } \partial B. \tag{7.3}$$

We claim that $\mu_2 = \mu_3$. Using polar coordinates (since $B \in R^2$), we write $u = u(r,\theta)$. Call $u_1 = u(r, \pi/2 - \theta)$; then $u_1 = \xi_1\varphi_1(r) + \xi_2\varphi_2(r, \pi/2-\theta) + \xi_2\varphi_3(r,\pi/2-\theta) + U_\xi = \xi_1\varphi_1(r) + \xi_2\varphi_3(r,\theta) + \xi_2\varphi_2(r,\theta) + U_\xi$, which implies that u_1 has the same 3 -signature as u. We claim that u_1 satisfies (7.3), with μ_2 and μ_3 switched. This will contradict the uniqueness of (μ_1, μ_2, μ_3), proving that $\mu_2 = \mu_3$. To prove the last claim, just observe

$$\begin{aligned}\Delta u_1 + bu_1^+ &= \mu_1\varphi_1(r) + \mu_2\varphi_2(r,\pi/2-\theta) + \mu_3\varphi_3(r,\pi/2-\theta) \\ &= \mu_1\varphi_1(r) + \mu_2\varphi_3(r,\theta) + \mu_3\varphi_2(r,\theta)\,.\end{aligned}$$

To recapitulate, for any (ξ_1, ξ_2) we can find (μ_1, μ_2) and $u(x)$ of the form (7.2) solving

$$\Delta u + bu^+ = \mu_1\varphi_1 + \mu_2\varphi_2 + \mu_2\varphi_3 \ \text{ for } x \in B, \ \ u = 0 \ \text{ on } \partial B\,. \tag{7.4}$$

It suffices to show that we can find a point $(\xi_1^0, \xi_2^0, \xi_2^0)$, with $\xi_2^0 \neq 0$, so that the corresponding $\mu_2^0 = 0$ and $\mu_1^0 \neq 0$ is arbitrary. Indeed, then $v = \frac{u}{\mu_1^0}$ solves (7.1), and gives us a non-radial solution.

We claim that μ_2 takes both negative and positive values at some points P and Q in the upper half of the (ξ_1, ξ_2, ξ_2) plane, i.e., when $\xi_2 > 0$. Assuming the claim for the moment, join P and Q by, say, a straight line. Since μ_2 changes continuously along that line, it must be zero somewhere; i.e., the problem (7.1) has a non-radial solution of the form (7.2), with $\xi_2 > 0$.

To prove the claim, we multiply (7.3) by φ_2, and integrate

$$-\lambda_2 \xi_2 + b \int_B u^+ \varphi_2 \, dx = \mu_2. \tag{7.5}$$

Write $u(x)$ in the form $u = s\,(a_1\varphi_1 + a_2\varphi_2 + a_2\varphi_3) + U_\xi$, with parameters $s > 0$, a_1, a_2 to be chosen, i.e., here $\xi_1 = sa_1$, $\xi_2 = \xi_3 = sa_2$. We claim that if a_1 and a_2 are fixed in such a way that

$$a_1\varphi_1 + a_2\varphi_2 + a_2\varphi_3 \quad \text{is positive or negative on } B, \tag{7.6}$$

then for $s > 0$ large the quantity on the left in (7.5) is asymptotic to

$$s\left[-\lambda_2 a_2 + b \int_B (a_1\varphi_1 + a_2\varphi_2 + a_2\varphi_3)^+ \varphi_2 \, dx\right]. \tag{7.7}$$

Indeed, consider the difference between the integrals in (7.5) and (7.7)

$$\begin{aligned} &\left| \int_B \left[(sa_1\varphi_1 + sa_2\varphi_2 + sa_2\varphi_3 + U_\xi)^+ - (sa_1\varphi_1 + sa_2\varphi_2 + sa_2\varphi_3)^+\right] \varphi_2 \, dx \right| \\ &\le \int_B |U_\xi||\varphi_2| \, dx \end{aligned}$$

since the function $f(x) = x^+$ is Lipschitz continuous, with Lipschitz constant one. The last integral is bounded uniformly in s, since $||U_\xi||_{L^2} \le c$ uniformly in sa_1, sa_2 and sa_3 that satisfy (7.6), by Lemma 2.2 (see Remark 1 after its proof).

We now fix $a_2 = 1$. If we now take $a_1 < 0$ and large in absolute value so that the integral in (7.7) is zero, we will get μ_2 to be negative. If we take $a_1 > 0$ and large, so that $a_1\varphi_1 + a_2\varphi_2 + a_2\varphi_3 > 0$ on B, we see that μ_2 is asymptotic to

$$s\left[-\lambda_2 + b \int_B (a_1\varphi_1 + \varphi_2 + \varphi_3)\, \varphi_2 \, dx\right] = s(b - \lambda_2) > 0\,.$$

By the above remarks we obtain a solution in the form (7.2), with $\xi_2 = s > 0$.

We obtain another solution by fixing $a_2 = -1$. Similarly to above, if we take $a_1 < 0$ and large in absolute value, we will get $\mu_2 > 0$. If we take $a_1 > 0$ and large, then $\mu_2 < 0$ for large s. We obtain a solution in the form (7.2), with $\xi_2 = -s < 0$.

The other two solutions are obtained in the form

$$u = \xi_1\varphi_1 + \xi_2\varphi_2 - \xi_2\varphi_3 + U_\xi, \tag{7.8}$$

with $\xi_2 \neq 0$. By Theorem 3.1 for any triple $(\xi_1, \xi_2, -\xi_2)$ we can find a *unique* triple (μ_1, μ_2, μ_3) and $u(x)$ solving

$$\Delta u + bu^+ = \mu_1\varphi_1 + \mu_2\varphi_2 + \mu_3\varphi_3 \ \text{ for } x \in B, \ \ u = 0 \ \text{ on } \partial B.$$

We claim that $\mu_2 = -\mu_3$. The proof is similar to the one above, this time setting $u_1 = u(r, 3\pi/2 - \theta)$. Then, exactly as above, we produce a solution of the form (7.8), with $\xi_2 > 0$, and another one with $\xi_2 < 0$, the third and the fourth non-radial solutions of the problem (7.1). $\square$

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