

## On Existence of Solutions for Two Classes of Nonlinear Problems

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## 1. Introduction.

We study existence of periodic solutions for nonlinear non-coercive boundary value problems of the form

$$\begin{aligned}
 (1.1) \quad & u_y = g(x, u, Du, D^2u, D^3u, D^4u) & y = 1 \\
 & \Delta u = \varepsilon f(x, y, u, Du, D^2u) & 0 < y < 1 \\
 & u = 0 & y = 0.
 \end{aligned}$$

Here  $f$  and  $g$  are  $2\pi$  periodic in each  $x_1, \dots, x_n$ ,  $\varepsilon$  is a small parameter, and we study existence of a  $2\pi$  periodic in each  $x_i$  solution  $u(x, y)$ .

Problems of the type (1.1) come up in applications, e.g., in the three-dimensional water wave theory, see M. Shinbrot [8]. We are interested in (1.1) primarily as a model non-coercive problem (i.e., the Lopatinski-Schapiro condition fails). In [3,4] we had considered the case of second order boundary conditions. Here we consider boundary conditions of the

fourth order, with generalization to an arbitrary order being quite transparent.

Let us assume for the discussion purposes that  $g = r(x) u_{x_1 x_1 x_1 x_1} + \rho(u_{x_1 x_1 x_1 x_1})$  with  $\rho(0) = \dot{\rho}(0) = 0$ . If one assumes  $r(x) \geq r_0 > 0$  for all  $x$ , then the problem can be solved using the Picard iterations in suitable Banach spaces (if  $f$  and  $g$  are smooth enough), see [5]. In case  $r(x) \geq 0$  there is a loss of derivatives on each iteration step, which we overcome using the Nash-Moser's method. The theorem 5.1 below applies in particular to the case  $r(x) \geq 0$ .

The main part of this paper consists in derivation of a priori estimates for the linearized problem, which is carried out in lemma 3.1, and which allows us to use the Nash-Moser technique. There are a number of versions of the Nash-Moser method, see e.g., L. Hormander [2] and K. Tso [9] among the most recent ones. We choose the one due to J. Schwartz [7], which appears to be one of the simplest and well-suited for the elliptic problems (we introduce a slight modification, which makes its application easier than in [3]).

In Section 6 we consider a perturbation problem on the torus  $T^n = [0, 2\pi]^n$ , of the type

$$u - \sum_{i,j=1}^n a_{ij}(x) u_{ij} = \epsilon f(x, u, Du, D^2 u),$$

where the operator on the left is assumed to be degenerate elliptic (in particular parabolic operators are allowed). We prove solvability for sufficiently small  $\epsilon$  by using the same version of the Nash-Moser technique. Our result extends the well-known work of P. Rabinowitz [11] and T. Kato [10], since we allow the perturbation term to depend on the derivatives of second order.

2. Notations and technical lemmas.

We consider functions of  $n + 1$  variables  $x = (x_1, \dots, x_n)$  and  $y$  which are  $2\pi$  periodic in each  $x_i$ ,  $i = 1, \dots, n$ . By  $V$  we denote the domain  $0 \leq x_i \leq 2\pi$ ,  $0 \leq y \leq 1$ ,  $i = 1, \dots, n$ ; by  $\partial V$  we denote its boundary, and  $V_t = \{(x, 1), 0 \leq x_i \leq 2\pi \text{ for all } i\}$ . We shall abbreviate  $\int_V f \equiv \int_V f dx dy$  and  $\int_t f \equiv \int_t f dx$ . We shall write  $\|\cdot\|_m$  for the norm in the Sobolev space  $H^m(V)$ ,  $\|\cdot\|_m^t$  for the one on  $H^m(V_t)$ . We write  $D$  or  $\nabla$  for the gradient and  $D^2$  for the Hessian in variables  $x, y$ ;  $D', \nabla', D'^2$  and also  $D'^3, D'^4$  for the same operations in the variable  $x$  only. We shall also need the norms (in  $V$  and on  $V_t$ )

$$|f|_N = \sum_{|\alpha| \leq N} |D^\alpha f|_{L^\infty}, \quad N = \text{integer} \geq 0.$$

All positive constants independent of the unknown functions we denote by  $c$ .

We need the following standard relations between our norms (see [3] for proofs and references).

Lemma 2.1. For any integer  $m \geq 0$  and any  $\epsilon > 0$  one can find a constant  $c(\epsilon)$  so that

- (i)  $\|\nabla v\|_m \leq \|v\|_{m+1}$
- (ii)  $\|v\|_m \leq \epsilon \|v\|_{m+1} + c(\epsilon) \|v\|_0$
- (iii)  $\|\nabla v\|_m \leq \epsilon \|v\|_{m+1} + c(\epsilon) \|v\|_0$ .

Lemma 2.2. Suppose  $f_1, f_2 \in C^r(V)$ ,  $r \geq 0$  is an integer. Then

- (i)  $\|f_1 f_2\|_r \leq c[\|f_1\|_0 \|f_2\|_r + \|f_2\|_0 \|f_1\|_r]$
- (ii)  $|f_1 f_2|_r \leq c[|f_1|_0 |f_2|_r + |f_2|_0 |f_1|_r]$ .

Obviously, similar inequalities are true for functions on  $V_t$ .

Lemma 2.3. Suppose  $w_1, \dots, w_s \in C^r(V)$ . Suppose that  $\phi = \phi(w_1, \dots, w_s)$  possesses continuous derivatives up to order  $r \geq 1$  bounded by  $c$  on

- $\max_i |w_i| < 1$ . Then
- (i)  $\|\phi(w_1, \dots, w_s)\|_r \leq c(\|w\|_r + 1)$
  - (ii)  $|\phi(w_1, \dots, w_s)|_r \leq c(|w|_r + 1)$ .

If in addition we assume

$$\phi(0, \dots, 0) = 0, \quad r \geq \left[\frac{n+1}{2}\right] + 1.$$

Then

$$\|\phi(w_1, \dots, w_s)\|_r = \delta(\|w\|_r),$$

where  $\delta(t) \rightarrow 0$  as  $t \rightarrow 0$ . (We denote  $\|w\|_r = \max_i \|w_i\|_r$ ). Similar conclusions

hold for functions on  $V_t$ .

**Lemma 2.4.** Let  $\ell, k, m$  be non-negative integers,  $k \leq m$ . Then

(i)  $\|u\|_{k+\ell} \leq c \|u\|_{m+\ell}^{k/m} \|u\|_{\ell}^{1-k/m}$

(ii)  $\overline{\|u\|}_{k+\ell} \leq c \overline{\|u\|}_{m+\ell}^{k/m} \overline{\|u\|}_{\ell}^{1-k/m}$ .

3. A priori estimates and existence in the linear case.

We start by deriving a priori estimates for the following non-coercive

in general problem.

**Lemma 3.1.** Consider the problem  $(x \in \mathbb{R}^n)$

$$\begin{aligned} u_y - \Delta' u + \sum_{|\alpha| \leq 4} r^\alpha(x) D'^\alpha u &= g(x) & y &= 1 \\ \Delta u - \varepsilon \sum_{|\alpha| \leq 2} c^\alpha(x, y) D^\alpha u &= f(x, y) & 0 < y < 1 \\ u &= 0 & y &= 0, \end{aligned} \quad (3.1)$$

with all the functions assumed to be  $2\pi$  periodic in each  $x_i$ ,  $i = 1, \dots, n$ .

Denote ( $k = \text{integer} \geq 0$ )  $r_k = \max_{|\alpha| \leq 4} |r^\alpha|_k$ ,  $c_k = \max_{|\alpha| \leq 2} |c^\alpha|_k$ ,  $\rho_k = r_k + c_k$ ,

$F_k = \|f\|_k + \|g\|_k$ . Also we denote  $p_4 = \rho_4$ ,  $p_5 = \rho_4 p_4 + \rho_5$ ,  $\dots$ ,  $p_k = \rho_4 p_{k-1}$

+  $\rho_5 p_{k-2} + \dots + \rho_{k-1} p_4 + \rho_k$  for  $k \geq 5$ .

Assume the following:

(ia) The fourth order terms have the following structure

$$\sum_{|\alpha|=4} r^\alpha(x) D'^\alpha u = \sum_{i,j=1}^n r_{ij}(x) u_{ijj},$$

$$\text{with } \sum_{i,j=1}^n r_{ij}(x) \xi_i \xi_j \geq 0 \text{ for all } x \text{ and } \xi_1, \dots, \xi_n.$$

(ib) In case  $n \geq 2$  the third order terms have the following structure

$$\sum_{|\alpha|=3} r^\alpha(x) D'^\alpha u = \sum_{i,j=1}^n \bar{r}_{ij}(x) u_{ijj}.$$

(If  $n \leq 2$  this assumption is automatically fulfilled.)

(ii)  $\rho_3 \leq \delta$ .

Then for  $\varepsilon$  and  $\delta$  sufficiently small the following a priori estimates hold

( $m = \text{integer}$ )

$$\|u\|_{m+1} + \overline{\|u\|}_{m+1} \leq c(F_m + p_4 F_{m-1} + p_5 F_{m-2} + \dots + p_m F_3), \quad m \geq 4$$

(3.2)

$$\|u\|_{m+1} + \overline{\|u\|}_{m+1} \leq c F_m, \quad m = 0, 1, 2, 3.$$

**Proof.** Step 1. Multiply the equation in (3.1) by  $u$  and integrate by

parts.

$$\begin{aligned} (3.3) \quad & -\int_t |\nabla u|^2 - \int_t |\nabla' u|^2 - \int_t \sum_{|\alpha| \leq 4} r_\alpha u D'^\alpha u + \int_t u g = \int_t f u \\ & + \varepsilon \int_t \sum_{|\alpha| \leq 2} c_\alpha u D^\alpha u; \\ & \int_t r_{ij,jj} u u_{ii} = \int_t r_{ij,jj} u_{ii} + \int_t r_{ij,jj} u u_{ii} + 2 \int_t r_{ij,jj} u_{ii}. \end{aligned}$$

The last two terms in the formula above can be absorbed into  $\int_t |\nabla' u|^2 + \int_t |\nabla u|^2$

as follows.

$$\begin{aligned} \int_t r_{ij,jj} u u_{ii} &= - \int_t r_{ij,jj} u_{ii} - \int_t r_{ij,jj} u_i^2; \\ \int_t r_{ij,j} u_j u_{ii} &= - \int_t r_{ij,j} u_j u_i + \frac{1}{2} \int_t r_{ij,jj} u_i^2. \end{aligned}$$

Corollary. If in addition we assume

$$\phi(0, \dots, 0) = 0, \quad r \geq \left[\frac{n+1}{2}\right] + 1.$$

Then

$$\|\phi(w_1, \dots, w_s)\|_r = \delta(\|w\|_r),$$

where  $\delta(t) \rightarrow 0$  as  $t \rightarrow 0$ . (We denote  $\|w\|_r = \max_1 \|w_1\|_r$ ). Similar conclusions

hold for functions on  $V_t$ .

Lemma 2.4. Let  $\ell, k, m$  be non-negative integers,  $k \leq m$ . Then

$$(i) \quad \|u\|_{k+\ell} \leq c \|u\|_{m+\ell}^{k/m} \|u\|_{\ell}^{1-k/m}$$

$$(ii) \quad \overline{\|u\|}_{k+\ell} \leq c \overline{\|u\|}_{m+\ell}^{k/m} \overline{\|u\|}_{\ell}^{1-k/m}.$$

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Lemma 3.1. Consider the problem ( $x \in R^n$ )

$$(3.1) \quad \begin{aligned} u_y - \Delta' u + \sum_{|\alpha| \leq 4} r^\alpha(x) D'^\alpha u &= g(x) & y &= 1 \\ \Delta u - \varepsilon \sum_{|\alpha| \leq 2} c^\alpha(x, y) D^\alpha u &= f(x, y) & 0 < y < 1 \\ u &= 0 & y &= 0, \end{aligned}$$

with all the functions assumed to be  $2\pi$  periodic in each  $x_i$ ,  $i = 1, \dots, n$ .

Denote ( $k = \text{integer} \geq 0$ )  $r_k = \max_{|\alpha| \leq k} |r^\alpha|_k$ ,  $c_k = \max_{|\alpha| \leq k} |c^\alpha|_k$ ,  $\rho_k = r_k + c_k$ ,

$F_k = \|f\|_k + \|g\|_k$ . Also we denote  $p_4 = \rho_4$ ,  $p_5 = \rho_4 p_4 + \rho_5$ , ...,  $p_k = \rho_4 p_{k-1} + \rho_5 p_{k-2} + \dots + \rho_{k-1} p_4 + \rho_k$  for  $k \geq 5$ .

Assume the following:

(ia) The fourth order terms have the following structure

$$\sum_{|\alpha|=4} r^\alpha(x) D'^\alpha u = \sum_{i,j=1}^n r_{ij}(x) u_{iijj},$$

$$\text{with } \sum_{i,j=1}^n r_{ij}(x) \xi_i \xi_j \geq 0 \text{ for all } x \text{ and } \xi_1, \dots, \xi_n.$$

(ib) In case  $n \geq 2$  the third order terms have the following structure

$$\sum_{|\alpha|=3} r^\alpha(x) D'^\alpha u = \sum_{i,j=1}^n \bar{r}_{ij}(x) u_{iijj}.$$

(If  $n \leq 2$  this assumption is automatically fulfilled.)

$$(ii) \quad \rho_3 \leq \delta.$$

Then for  $\varepsilon$  and  $\delta$  sufficiently small the following a priori estimates hold

( $m = \text{integer}$ )

$$\|u\|_{m+1} + \overline{\|u\|}_{m+1} \leq c(F_m + p_4 F_{m-1} + p_5 F_{m-2} + \dots + p_m F_3), \quad m \geq 4$$

(3.2)

$$\|u\|_{m+1} + \overline{\|u\|}_{m+1} \leq c F_m, \quad m = 0, 1, 2, 3.$$

Proof. Step 1. Multiply the equation in (3.1) by  $u$  and integrate by

parts.

$$(3.3) \quad \begin{aligned} -\int |\nabla u|^2 - \int_t |\nabla' u|^2 - \int_t \sum_{|\alpha| \leq 4} r_\alpha u D'^\alpha u + \int_t u g &= \int_t f u \\ &+ \varepsilon \int_t \sum_{|\alpha| \leq 2} c_\alpha u D^\alpha u; \\ \int_t r_{ij,jj} u u_{iijj} &= \int_t r_{ij,jj} u_{jj} u_{ii} + \int_t r_{ij,jj} u u_{ii} + 2 \int_t r_{ij,jj} u_{ji} u_{ii}. \end{aligned}$$

The last two terms in the formula above can be absorbed into  $\int_t |\nabla' u|^2 + \int_t |\nabla u|^2$

as follows.

$$\begin{aligned} \int_t r_{ij,jj} u u_{iijj} &= - \int_t r_{ij,jj} u_{ii} u_{jj} - \int_t r_{ij,jj} u_{ii}^2; \\ \int_t r_{ij,j} u_{ji} u_{ii} &= - \int_t r_{ij,j} u_{ji} u_{ii} + \frac{1}{2} \int_t r_{ij,jj} u_{ii}^2. \end{aligned}$$

Treating the terms corresponding to  $|\alpha| = 3$  and  $|\alpha| = 2$  in a similar fashion and using lemma 2.1, we easily derive from (3.3)

$$(3.4) \quad \int_t |\nabla u|^2 + \int_t |\nabla' u|^2 \leq c(\varepsilon \|u\|_2^2 + \int_t f^2 + \int_t g^2).$$

Step 2. Multiply the equation in (3.1) by  $u_{kk}$  and integrate by parts ( $1 \leq k \leq n$ ).

$$(3.5) \quad \begin{aligned} \int_t |\nabla u_k|^2 + \int_t |\nabla' u_k|^2 - \int_t \sum_{|\alpha| \leq 4} r_{\alpha} u_{kk} D^{\alpha} u + \int_t u_{kk} g \\ = \int_t f u_{kk} + \varepsilon \int_t \sum_{|\alpha| \leq 2} c^{\alpha} u_{kk} D^{\alpha} u. \end{aligned}$$

After repeated integration by parts

$$\begin{aligned} \int_t r_{ij} u_{kk} u_{iijj} &= \int_t r_{ij} u_{kkjj} u_{ii} + \int_t r_{ij,jj} u_{kk} u_{ii} \\ &\quad + 2 \int_t r_{ij,j} u_{kkj} u_{ii} \\ &= - \int_t r_{ij} u_{kkjj} u_{kii} - \int_t r_{ij,j} u_{kkj} u_{ii} + \int_t r_{ij,jj} u_{kk} u_{ii} \\ &\quad + 2 \int_t r_{ij,j} u_{kkj} u_{ii}. \end{aligned}$$

The second, third and fourth terms on the right after repeated integration by parts are easily absorbed into  $\sum_{k=1}^n \int_t |\nabla' u_k|^2$ .

From the equation (3.1) we estimate

$$\int u_{yy}^2 \leq c \left( \sum_{k=1}^n \int_t |\nabla u_k|^2 + \int_t f^2 \right).$$

Combining this with the estimate (3.5), summing in  $k$ , we get

$$\|u\|_2 + \|\overline{u}\|_2 \leq c(\varepsilon \|u\|_2 + \|f\|_0 + \|g\|_0),$$

from which our estimate (3.2) follows with  $m = 0$ .

Step 3. Let  $\beta = (\beta_1, \dots, \beta_n, 0)$ ,  $|\beta| \leq m$ . Differentiate (3.1) and denote  $u^{\beta} = D^{\beta} u$ ,  $e_p$ -coordinate vectors in  $R^n$

$$(3.6) \quad \begin{aligned} u_y^{\beta} - \Delta' u^{\beta} + \sum_{|\alpha| \leq 4} r^{\alpha} D^{\alpha} u^{\beta} + \sum_{p=1}^n c_p \sum_{|\alpha| \leq 4} r_p^{\alpha} D^{\alpha} u^{\beta-e_p} + \dots = g^{\beta} \quad y = 1, \\ \Delta u^{\beta} - \varepsilon \sum_{|\alpha| \leq 2} c^{\alpha} D^{\alpha} u^{\beta} - \dots = f^{\beta} \quad 0 < y < 1, \\ u^{\beta} = 0 \quad y = 0. \end{aligned}$$

Multiply (3.6) by  $u^{\beta}$  and integrate by parts,

$$(3.7) \quad \begin{aligned} - \int_t |\nabla u^{\beta}|^2 - \int_t |\nabla' u^{\beta}|^2 - \sum_{|\alpha| \leq 4} \int_t r^{\alpha} u^{\beta} D^{\alpha} u^{\beta} - \\ - \sum_{p=1}^n c_p \sum_{|\alpha| \leq 4} \int_t r_p^{\alpha} u^{\beta} D^{\alpha} u^{\beta-e_p} - \dots + \int_t u^{\beta} g^{\beta} = \int_t u^{\beta} f^{\beta} + \\ + \varepsilon \sum_{|\alpha| \leq 2} \int_t c^{\alpha} u^{\beta} D^{\alpha} u^{\beta} + \varepsilon \sum_{p=1}^n c_p \sum_{|\alpha| \leq 2} \int_t c_p^{\alpha} u^{\beta} D^{\alpha} u^{\beta-e_p} + \dots \end{aligned}$$

Similarly to the Step 1

$$(3.8) \quad \begin{aligned} \int_t r_{ij} u^{\beta} u^{\beta}_{iijj} &= \int_t r_{ij} u^{\beta}_{jj} u^{\beta}_{ii} + \int_t r_{ij,jj} u^{\beta} u^{\beta}_{ii} \\ &\quad + 2 \int_t r_{ij,j} u^{\beta} u^{\beta}_{jii}. \end{aligned}$$

Denote  $\beta - e_p = \gamma$ ,  $|\gamma| = m - 1$ . Then for  $|\alpha| = 4$ ,

$$(3.9) \quad \begin{aligned} \int_t r_p^{\alpha} u^{\beta} D^{\alpha} u^{\beta-e_p} &= \int_t r_{ij,p} u_p^{\gamma} u^{\gamma}_{iijj} \\ &= \int_t r_{ij,pjj} u_p^{\gamma} u^{\gamma}_{ii} + \int_t r_{ij,p} u_{pjj}^{\gamma} u^{\gamma}_{ii} \\ &\quad + 2 \int_t r_{ij,pj} u_p^{\gamma} u^{\gamma}_{jii}. \end{aligned}$$

One easily sees that all terms on the right in (3.8) and (3.9) can be absorbed into the first two terms on the left in (3.7). Similarly the third and the

fourth terms on the left in (3.7) as well as the first four terms on the right in (3.7) are absorbed into the first two terms on the left in (3.7). So that

$$(3.10) \quad \int_t |\nabla u^\beta|^2 + \int_t |\nabla' u^\beta|^2 \leq c(F_m^2 + r_4^2 \|\bar{u}\|_m^2 + \dots + r_m^2 \|\bar{u}\|_4^2 + c_4^2 \|u\|_{m-2}^2 + \dots + c_m^2 \|u\|_2^2).$$

Multiply (3.6) by  $u_{kk}^\beta$ , integrate by parts and sum in  $k = 1, \dots, n$ .

Proceeding similarly, we estimate

$$(3.11) \quad \int |\nabla u_k^\beta|^2 + \int_t |\nabla' u_k^\beta|^2 \leq c(\|u\|_{m+2}^2 + F_m^2 + r_4^2 \|\bar{u}\|_m^2 + \dots + r_m^2 \|\bar{u}\|_4^2 + c_4^2 \|u\|_{m-2}^2 + \dots + c_m^2 \|u\|_2^2).$$

Summing in all  $\beta$  with  $|\beta| \leq m$ ,  $m \geq 4$ , and estimating all the missing derivatives from the equation (3.1), we obtain

$$\|u\|_{m+2} + \|\bar{u}\|_{m+2} \leq c(F_m + \rho_4(\|u\|_m + \|\bar{u}\|_m) + \rho_5(\|u\|_{m-1} + \|\bar{u}\|_{m-1}) + \dots + \rho_m(\|u\|_4 + \|\bar{u}\|_4)),$$

from which (3.2) easily follows.

**Lemma 3.2.** Consider the problem (3.1). Assume that conditions (i) and (ii) of lemma 3.1 are satisfied;  $f \in H^m(V)$ ,  $g \in H^m(V_t)$ ,  $\rho_m \leq c$  for  $m \geq [\frac{n}{2}] + 4$ . Then for  $\varepsilon$  and  $\delta$  sufficiently small the problem (3.1) has a unique solution of class  $H^{m+2}(V)$ .

**Proof.** One easily verifies that the problem ( $\sigma > 0$ )

$$(3.12) \quad \begin{aligned} u_y + \sigma \Delta'^2 u - t \Delta' u + t \sum_{|\alpha| \leq 4} r^\alpha D^\alpha u &= g(x) & y = 1, \\ \Delta u - \varepsilon \sum_{|\alpha| \leq 2} c^\alpha D^\alpha u &= f(x, y) & 0 < y < 1, \\ u &= 0 & y = 0, \end{aligned}$$

is coercive at  $y = 1$  for  $\sigma > 0$ ,  $0 \leq t \leq 1$  and  $\varepsilon$  small, i.e., the Lopatinski-Shapiro condition is satisfied, and that the estimate (3.2) holds

with  $c$  independent of  $\sigma$ . To see coercivity one follows the proof of lemma 5.1 in [3], throwing out lower order terms and freezing all  $r^\alpha$ ,  $c^\alpha$  at some  $(x_0, 1) \in V_t$ , and then taking the Fourier transform  $v(\xi) = \frac{1}{(2\pi)^{n/2}} \int e^{ix \cdot \xi} u(x) dx$ .

One notices that the Fourier transform of the boundary condition at  $y = 1$  is

$$(\sigma |\xi|^4 + t |\xi|^2 + t \sum_{i,j=1} r_{ij}(x_0) \xi_i^2 \xi_j^2) v,$$

with the bracket being positive by our assumptions. The rest of the argument is exactly as in lemma 5.1 of [3].

Since the problem (3.12) is coercive it defines a Noether map from  $H^{m+2}(V)$  into  $H^m(V) \times H^{m-5/2}(V_t)$ , whose index is invariant of homotopy transformations which do not take the problem out of the coercive class. Letting  $t, \varepsilon \rightarrow 0$  we see that the index of (3.12) is the same as that of

$$(3.13) \quad \begin{aligned} u_y + \sigma \Delta'^2 u &= g & y = 1, \\ \Delta u &= f & 0 < y < 1, \\ u &= 0 & y = 0. \end{aligned}$$

By an elementary Fourier analysis one sees that the problem (3.13) is uniquely solvable, so that its index is zero (the general elliptic theory applies to (3.13)). Since the estimate (3.2) implies uniqueness and (3.13) has index 0, it follows that the problem (3.13) is solvable. Denote its solution for  $t = 1$  by  $v^\sigma$ . Since  $\|v^\sigma\|_{m+2} \leq c$  it follows that  $\{v^\sigma\}$  is precompact in  $H^{m+1}(V)$ , and hence it converges to some  $v \in H^{m+1}(V)$  as  $\sigma \rightarrow 0$  along some sequence. Passing to the limit in (3.12) as  $t = 1$  and  $\sigma \rightarrow 0$ , we see that  $v$  is solution of (3.1). By lemma 3.1,  $v \in H^{m+2}(V)$ , completing the proof.

#### 4. Nash's Implicit Function Theorem.

We start by introducing a standard concept of a scale of Banach spaces.

**Definition 1.** Suppose we have a family of Banach spaces  $B^m$  indexed by a parameter  $m \geq 0$ . We say that this family forms a Banach scale if  $B^m \subset B^{m_1}$  for  $m > m_1$ , and  $\|u\|_{m_1} \leq \|u\|_m$  for  $u \in B^m$ .

**Definition 2.** We call  $B^m$  a Banach scale with smoothing if there exists a family of smoothing operators  $S(t)$ , depending on a parameter  $t \geq 0$ , with the properties ( $0 \leq r \leq \rho$ ,  $I$ -identity operator)

$$\begin{aligned} (S_1) \quad \|S(t)u\|_\rho &\leq ct^{\rho-r} \|u\|_r, \quad u \in B^r \\ (S_2) \quad \|(I-S(t))u\|_r &\leq ct^{r-\rho} \|u\|_r, \quad u \in B^\rho \end{aligned}$$

It is well known that  $H^m(V)$  is a Banach scale with smoothing.

The following theorem is a slight modification of J. Schwartz's form of the Nash's implicit function theorem, see [7]. The proof in [7] contained some errors, which turned out to be easily correctable along the same lines.

**Theorem 4.1.** Let  $B_1^m, B_2^m$  be two Banach scales, the first one with smoothing. Let  $F(u): B_1^m \rightarrow B_2^{m-\alpha}$  ( $0 \leq \alpha \leq m$ ) be a (non-linear) operator with the domain  $D(F) = \{u \in B_1^m, \|u\|_m < \delta, \delta > 0\}$ . Suppose that

- (i)  $F(u)$  has two continuous Frechet derivatives both bounded by  $c$ .
- (ii) There exists a map  $L(u)$  with domain  $D(L) = D(F)$  and range in the space  $B(B_2^{m-\alpha}, B_1^{m-\alpha})$  of bounded linear operators on  $B_2^{m-\alpha}$  to  $B_1^{m-\alpha}$  such that

$$\begin{aligned} \text{(iia)} \quad F'(u)L(u)h &= h, \quad h \in B_2^{m-\alpha}, \quad u \in D(F) \\ \text{(iib)} \quad \|L(u)h\|_{m-\alpha} &\leq c\|h\|_{m-\alpha}, \quad h \in B_2^{m-\alpha}, \quad u \in D(F) \\ \text{(iic)} \quad \|L(u)F(u)\|_{m+8\alpha} &\leq c(1+\|u\|_{m+9\alpha}), \quad u \in B_1^{m+9\alpha} \cap D(F). \end{aligned}$$

Then if  $\|F(0)\|_{m-\alpha}$  is small enough,  $F(D(F))$  contains the origin.

**Proof.** Let  $\kappa = \frac{4}{3}$ ,  $\beta, \mu, \nu$  positive constants to be specified later.

Set  $u_0 = 0$  and define inductively

$$(4.1) \quad u_{n+1} = u_n - S_n L(u_n) F(u_n), \quad S_n = S(e^{\beta \kappa^n})$$

We will prove inductively that

$$(4.2.n) \quad u_n \in D(F) \cap B_1^{m+9\alpha}$$

$$(4.3.n) \quad \|u_{n+1} - u_n\|_m \leq e^{-\mu \alpha \beta \kappa^{n+1}}$$

$$(4.4.n) \quad 1 + \|u_n\|_{m+9\alpha} \leq e^{\nu \alpha \beta \kappa^n}$$

Assume that (4.2.j), (4.3.j), (4.4.j) are true for  $0 \leq j \leq n$ . Estimate

$$\begin{aligned} \|u_{n+1}\|_m &\leq \sum_{j=0}^n \|u_{j+1} - u_j\|_m \leq \sum_{j=0}^n e^{-\mu \alpha \beta \kappa^{j+1}} \\ &\leq \sum_{j=0}^{\infty} e^{-\mu \alpha \beta (\kappa-1)(j+1)} = \frac{e^{-\mu \alpha \beta (\kappa-1)}}{1 - e^{-\mu \alpha \beta (\kappa-1)}} < \delta, \end{aligned}$$

if  $\beta$  is chosen sufficiently large. This gives the first part of (4.2.n+1).

Next

$$\begin{aligned} (4.5) \quad \|u_{n+2} - u_{n+1}\|_m &= \|S_{n+1} L(u_{n+1}) F(u_{n+1})\|_m \\ &\leq c e^{\alpha \beta \kappa^{n+1}} \|L(u_{n+1}) F(u_{n+1})\|_{m-\alpha} \leq c^2 e^{\alpha \beta \kappa^{n+1}} \|F(u_{n+1})\|_{m-\alpha} \\ &\leq c^2 e^{\alpha \beta \kappa^{n+1}} \|F(u_n) - F'(u_n) S_n L(u_n) F(u_n)\|_{m-\alpha} \\ &\quad + c^3 e^{\alpha \beta \kappa^{n+1}} e^{-2\mu \alpha \beta \kappa^{n+1}} \end{aligned}$$

For the last step the Taylor series with Lagrange remainder was used.

Estimate

$$\begin{aligned} \|F(u_n) - F'(u_n) S_n L(u_n) F(u_n)\|_{m-\alpha} &= \|F'(u_n) (I - S_n) L(u_n) F(u_n)\|_{m-\alpha} \\ &\leq c^2 e^{-8\alpha \beta \kappa^n} \|L(u_n) F(u_n)\|_{m+8\alpha} \\ &\leq c^2 e^{-8\alpha \beta \kappa^n} (1 + \|u_n\|_{m+9\alpha}) \leq c^2 e^{-8\alpha \beta \kappa^n} e^{\nu \alpha \beta \kappa^n}. \end{aligned}$$

Hence (fix  $c > 1$ )

$$\begin{aligned} \|u_{n+2} - u_{n+1}\|_m &\leq c^5 (\exp \alpha \beta \kappa^n (\kappa-8+\nu) + \exp \alpha \beta \kappa^{n+1} (1-2\mu)) \\ &\leq e^{-\mu \alpha \beta \kappa^{n+2}}, \end{aligned}$$

provided  $\beta$  is large enough and

$$(4.6) \quad \kappa - 8 + \nu < -\mu\kappa^2, \quad 1 - 2\mu < -\mu\kappa.$$

Next

$$\begin{aligned} 1 + \|u_{n+1}\|_{m+9\alpha} &\leq 1 + \sum_{j=0}^n \|S_j L(u_j) F(u_j)\|_{m+9\alpha} \\ &\leq 1 + c \sum_{j=0}^n e^{\alpha\beta\kappa^j} \|L(u_j) F(u_j)\|_{m+8\alpha} \\ &\leq 1 + c^2 \sum_{j=0}^n e^{\alpha\beta\kappa^j} (1 + \|u_j\|_{m+9\alpha}) \\ &\leq 1 + c^2 \sum_{j=0}^n e^{\alpha\beta(1+\nu)\kappa^j} \leq e^{\nu\alpha\beta\kappa^{n+1}} \end{aligned}$$

provided  $\beta$  is large enough and

$$(4.7) \quad 1 + \nu < \kappa\nu.$$

The induction step is now complete once we notice that one can pick  $\mu > \frac{3}{2}$ ,  $\mu \approx \frac{3}{2}$ ,  $\nu > 3$ ,  $\nu \approx 3$ ,  $\kappa = \frac{4}{3}$  satisfying both (4.6) and (4.7). To prove (4.3.0) we write

$$\begin{aligned} \|u_1 - u_0\|_m &= \|S_0 L(0) F(0)\|_m \leq ce^{\beta\alpha} \|L(0) F(0)\|_{m-\alpha} \\ &\leq ce^{\beta\alpha} \|F(0)\|_{m-\alpha} \leq e^{-\mu\alpha\beta\kappa} \end{aligned}$$

assuming  $\|F(0)\|_{m-\alpha}$  is chosen small enough (after fixing all the constants).

With (4.4.0) holding if  $\beta$  is large enough, the proof is now complete.

## 5. Existence for the Nonlinear Problem.

We are ready now for the main existence result. Laplacians in both equation and the boundary condition can be easily replaced by general second

order uniformly elliptic operators. The crucial condition (i) comes from the lemma 3.1. It ensures positivity of the linearized operator at the boundary. This hypothesis appears to be natural since it excludes the possibility of small divisors (see [3]).

Theorem 5.1. Consider the problem

$$(5.1) \quad \begin{aligned} u_y - \Delta' u &= g(x, u, D'u, D'^2 u, D'^3 u, D'^4 u) & y &= 1, \\ \Delta u &= \epsilon f(x, y, u, Du, D^2 u) & 0 < y < 1, \\ u &= 0 & y &= 0. \end{aligned}$$

Denote  $r^\alpha = -\frac{\partial g}{\partial D^\alpha u}$  and  $c^\alpha = -\frac{\partial f}{\partial D^\alpha u}$ ; and assume that

(i)  $r^\alpha$  have the same structure as in the conditions (ia) and (ib) of

lemma 3.1. Moreover, they vanish identically at  $u = 0$  for all  $|\alpha| \leq 4$ .

(ii)  $g(x, 0, \dots, 0) \equiv 0$ ,  $f(x, y, 0, \dots, 0) \equiv 0$ , and both functions are  $2\pi$  periodic in each  $x_i$ ,  $i = 1, \dots, n$ .

(iii)  $f, g \in C^{m_0}$  in all variables, with  $m_0 = 10[\frac{n}{2}] + 58$ .

Then for  $\epsilon$  sufficiently small the problem (5.1) has a  $2\pi$  periodic in each  $x_i$  solution of class  $C^4(\bar{V})$ .

Proof. Define an operator  $F(u): B_1^m \rightarrow B_2^{m-\alpha}$  as follows

$$F(u) = \begin{pmatrix} u_y - \Delta' u - g(x, u, D'u, D'^2 u, D'^3 u, D'^4 u) \\ \Delta u - \epsilon f(x, y, u, Du, D^2 u) \end{pmatrix}$$

with the Banach scales ( $B_1^m$  is known to be a scale with smoothing)

$$B_1^m = \{u \in H^m(V), \|u\|_m \leq \delta \text{ and } u(x, 0) \equiv 0\},$$

$$B_2^{m-\alpha} = H^{m-\alpha}(V) \times H^{m-\alpha}(V_t) \text{ with the norm } \|\cdot\|_{m-\alpha} + \|\cdot\|_{m-\alpha}.$$



Here  $m = 2[\frac{n}{2}] + 10$ ,  $\alpha = [\frac{n}{2}] + 6$ , and the constant  $\delta$  will be specified later.

We shall get solution of our problem by solving

$$(5.2) \quad F(u) = 0.$$

For this we verify the conditions of the theorem 4.1. Compute

$$F'(u)v = \begin{bmatrix} v_y - \Delta'v + \sum_{|\alpha| \leq 4} r^\alpha D^\alpha v \\ \Delta v - \varepsilon \sum_{|\alpha| \leq 2} c^\alpha D^\alpha v \end{bmatrix}$$

By lemma 2.3 and the Sobolev imbedding we estimate

$$(5.3) \quad \begin{aligned} r_{m-\alpha} &\leq \max_{\alpha} \|r^\alpha\|_{m-\alpha+[\frac{n}{2}]+1} \leq c(\|u\|_{m-\alpha+[\frac{n}{2}]+6} + 1) \leq c(\delta+1) \leq c, \\ c_{m-\alpha} &\leq \max_{\alpha} \|c^\alpha\|_{m-\alpha+[\frac{n+1}{2}]+1} \leq c(\|u\|_{m-\alpha+[\frac{n}{2}]+4} + 1) \leq c, \\ r_3 &\leq \max_{\alpha} \|r^\alpha\|_{[n/2]+4} = \delta(\|u\|_m), \text{ where } \delta(t) \rightarrow 0 \text{ as } t \rightarrow 0. \end{aligned}$$

So that for  $u \in B_1^m$

$$\|F'(u)\| = \sup_{\|v\|_m \leq 1} (\|v_y - \Delta'v + \sum_{|\alpha| \leq 4} r^\alpha D^\alpha v\|_{m-\alpha} + \|\Delta v - \varepsilon \sum_{|\alpha| \leq 2} c^\alpha D^\alpha v\|_{m-\alpha}) \leq c,$$

and the boundness of  $F''(u)$  is proved similarly. In the same way one checks that

$$\sup_{\|v\|_m \leq 1} \|(F'(u) - F'(w))v\|_{m-\alpha} \leq \varepsilon \text{ if } \|u-w\|_m < \delta(\varepsilon),$$

which implies continuity of  $F'(u)$ . The continuity of  $F''(u)$  is proved similarly.

Conditions (iia) and (iib) of the theorem 4.1 follow from lemmas 3.2 and 3.1 respectively in view of the estimates (5.3). It remains to check (iic). By lemma 3.1

$$(5.4) \quad \|L(u)F(u)\|_{m+8\alpha} \leq c(F_{m+8\alpha-1} + p_4 F_{m+8\alpha-2} + \dots + p_{m+8\alpha-1} F_3).$$

Denote  $\tau = \|u\|_{m+9\alpha}^{\frac{1}{m+9\alpha-[\frac{n}{2}]-9}}$ . By lemma 2.4 we have ( $k = [n/2] + 10, \dots, m+9\alpha-1$ )

$$\|u\|_k \leq c\|u\|_{m+9\alpha}^{\frac{k-[\frac{n}{2}]-9}{m+9\alpha-[\frac{n}{2}]-9}} \|u\|_{[\frac{n}{2}]+9}^{1-\frac{k-[\frac{n}{2}]-9}{m+9\alpha-[\frac{n}{2}]-9}} \leq c\tau^{k-[\frac{n}{2}]-9}$$

By (5.3) we have for  $k = 4, \dots, m+8\alpha-1$

$$\rho_k \leq c(\|u\|_{k+[\frac{n}{2}]+6} + 1) \leq c(\tau^{k-3} + 1).$$

Then we obtain inductively

$$p_4 = \rho_4 \leq c(\tau+1), \dots, p_{m+8\alpha-1} \leq c(\tau^{m+8\alpha-4} + 1).$$

$$F_k \leq c(\|u\|_{k+5} + 1) \leq c(\tau^{k-[\frac{n}{2}]-4} + 1), \quad k = m-4, \dots, m+8\alpha-1,$$

$$F_k \leq c, \quad k = 1, \dots, m-5.$$

Using these estimates in (5.4) we get

$$\|L(u)F(u)\|_{m+8\alpha} \leq c(\tau^{m+8\alpha-4} + 1) \leq c(\|u\|_{m+9\alpha} + 1),$$

since  $m + 8\alpha - 4 \leq m + 9\alpha - [n/2] - 9$  by choice of  $\alpha$ . So that by fixing  $m$  and  $\alpha$  as above,  $m_0 = m + 8\alpha$ , and  $\varepsilon, \delta$  small enough, we complete the proof.

## 6. A Singular Perturbation Problem.

We show that similar techniques produce existence results for a class of singular perturbation problems. First we need two lemmas, analogous to the ones of Section 3. We denote  $T = T^n = [0, 2\pi]^n$ , the  $n$ -torus;  $\int_T f = \int_T f(x) dx$ .

Lemma 6.1. Consider the problem

$$(6.1) \quad u(x) - a_{ij}(x)u_{ij}(x) - a_i(x)u_i - a_0(x)u = f(x),$$

where all functions are  $2\pi$  periodic in each  $x_i$ ,  $i = 1, \dots, n$  (i.e.,  $x \in T$ ),  $a_{ij}(x)$  is a symmetric matrix, and the summation convention is used throughout this section. Assume that

$$a_{ij}(x)\xi_i\xi_j \geq 0 \quad \text{for all } x \in T, \quad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

For integer  $k \geq 0$  denote

$$a^k = \max_{i, \ell, m} (|a_0|_k, |a_i|_k, |a_{\ell m}|_k),$$

$$p_3 = a^3, \quad p_n = a^3 p_{n-1} + a^4 p_{n-2} + \dots + a^{n-1} p_3 + a^n \quad \text{for } n \geq 4.$$

Then for  $a^2$  sufficiently small the following estimates hold ( $m = \text{integer}$ )

$$(6.2) \quad \|u\|_m \leq c(\|f\|_m + p_3\|f\|_{m-1} + p_4\|f\|_{m-2} + \dots + p_m\|f\|_2), \quad m > 2$$

$$\|u\|_m \leq c\|f\|_m \quad \text{for } m = 0, 1, 2.$$

Proof. To simplify notation assume that  $a_i(x) = a_0(x) \equiv 0$  for all  $i$ .

Multiply (6.1) by  $u$  and integrate by parts

$$\int u^2 + \int a_{ij}u_iu_j - \frac{1}{2} \int a_{ij,ij}u^2 = \int fu,$$

which implies (6.2) for  $m = 0$ .

Let now  $u^\alpha \equiv D^\alpha u$  with  $|\alpha| \leq m$ . Then

$$(6.3) \quad u^\alpha - a_{ij}u_{ij}^\alpha - a_{ij}^{1\alpha-1}u_i - a_{ij}^{2\alpha-2}u_{ij} - a_{ij}^{3\alpha-3}u_{ij} - \dots - a_{ij}^\alpha u_{ij} = f^\alpha,$$

where we denote  $a_{ij}^{k\alpha-k} = \sum_{|\gamma|=k} c_\gamma D^\gamma a_{ij} u_{ij}^{\alpha-\gamma}$ ,  $c_\gamma$  the coefficients from the Leibnitz rule. Multiply (6.3) by  $u^\alpha$  and integrate over  $T$

$$(6.4) \quad \int (u^\alpha)^2 - \int a_{ij}u_{ij}^\alpha u^\alpha - \int a_{ij}^{1\alpha-1}u_i^\alpha u^\alpha - \dots - \int a_{ij}^\alpha u_{ij}^\alpha u^\alpha = \int f^\alpha u^\alpha.$$

Integrating by parts

$$- \int a_{ij}^{1\alpha-1}u_i^\alpha u^\alpha = \int a_{ij}^{1\alpha-1}u_i^\alpha u_j^\alpha - \frac{1}{2} \int a_{ij,ij} (u^\alpha)^2.$$

Notice

$$(6.5) \quad \int a_{ij}^{1\alpha-1}u_i^\alpha u_j^\alpha = \sum_{k=1}^n c_k \int a_{ij,k}^{1\alpha-1} u_i^{\alpha-e_k} u_j^{\alpha-e_k}.$$

Consider the integral  $I = \int a_{ij,k}^{1\alpha-1} u_i^{\alpha-e_k} u_j^{\alpha-e_k}$ . We integrate by parts taking the derivatives  $\frac{\partial}{\partial x_i}$  and  $\frac{\partial}{\partial x_j}$  off the second factor, and the derivative  $\frac{\partial}{\partial x_k}$  off the third. We obtain

$$I = -I + \dots,$$

where all terms not shown on the right have  $a_{ij}$  differentiated exactly twice. Solving for  $I$  we see that all terms on the right in (6.5) are easily absorbed into  $\sum_{|\alpha|=n} \int (u^\alpha)^2$ . Summing in  $\alpha$  we obtain as in lemma 3.1 (for  $m > 2$ )

$$\|u\|_m \leq c(\|f\|_m + a^3\|u\|_{m-1} + a^4\|u\|_{m-2} + \dots + a^m\|u\|_2),$$

and the proof follows.

Lemma 6.2. Assume all conditions of the lemma 6.1, and that  $a^m \leq c$ ,  $f \in H^m(T)$ ,  $m \geq \max(\lfloor \frac{n}{2} \rfloor + 1, 3)$ . Then for  $a^2$  sufficiently small the problem (6.1) has a unique solution of class  $H^m(T)$ .

Proof. Consider  $(x \in T^n, \sigma = \text{const} > 0)$

$$(6.6) \quad u - \varepsilon a_{ij} u_{ij} - \varepsilon a_i u_i - \varepsilon a_0 u - \sigma \Delta u = f.$$

This is a uniformly elliptic equation on  $T^n$ , so its index (as an operator from  $H^{m+2}(T)$  to  $H^m(T)$ ) is defined and homotopy invariant. By letting  $\varepsilon \rightarrow 0$ , we get an equation

$$u - \sigma \Delta u = f,$$

whose index is zero, as follows by a simple Fourier analysis. One easily sees that the estimates (6.2) hold for (6.6) with  $c$  independent of  $\sigma$ , and hence (6.6) can have at most one solution. Since the index of (6.6) is zero, it has a solution  $u^\sigma \in H^{m+2}(T)$ , and by (6.2)

$$\|u^\sigma\|_m \leq c, \text{ uniformly in } \sigma > 0.$$

If  $\sigma \rightarrow 0$  along a sequence, passing to a subsequence we get  $u^\sigma \rightarrow u \in H^{m-1}(T)$ , where  $u$  is a solution of (6.1). Applying (6.2) again, we conclude  $u \in H^m(T)$ .

Theorem 6.1. On the torus  $T^n$  consider the problem

$$(6.7) \quad u = f(x, u, Du, D^2u).$$

Here  $f = f_1(x, u, Du, D^2u) + \varepsilon f_2(x, u, Du, D^2u)$  with  $f_1(x, 0, 0, 0) = f_{1u}(x, 0, 0, 0) = f_{1u_i}(x, 0, 0, 0) = f_{1u_{k\ell}}(x, 0, 0, 0) = 0$  for all  $i, k, \ell = 1, \dots, n$ , and both functions are assumed to be  $2\pi$  periodic in each  $x_i$ ,  $i = 1, \dots, n$ . Denote  $a_{ij} = \frac{\partial f}{\partial u_{ij}}, a_i = \frac{\partial f}{\partial u_i}, a_0 = \frac{\partial f}{\partial u}$ . For  $x \in T$  and other variables sufficiently small in absolute values, assume that the conditions (i) of (ii) of lemma 6.1 are satisfied, and  $f \in C^{m_0}$  with  $m_0 = \max([\frac{n}{2}] + 1, 3) + 9[\frac{n}{2}] + 27$ . Then for  $\varepsilon$  sufficiently small the problem (6.1) has a  $2\pi$  periodic in each  $x_i$  solution of class  $C^2(T)$ .

Proof. Define  $F(u) = u - f(x, u, Du, D^2u)$  and consider it as a map  $F: B^m(T) \rightarrow H^{m-\alpha}(T)$ , where  $B^m = \{u \in H^m(T): \|u\|_m \leq \delta\}$ , with constant  $\delta > 0$  and

integers  $m \geq \alpha$  to be specified. We shall solve  $F(u) = 0$  by applying the theorem 4.1. Compute

$$F'(u)v = v - f_{u_{ij}} v_{ij} - f_{u_i} v_i - f_u v,$$

$$F''(u)(v, w) = -f_{u_{ij}u_{k\ell}} v_{ij} w_{k\ell} - f_{u_{ij}u_k} v_{ij} w_k - \dots$$

Assuming  $m - \alpha \geq [\frac{n}{2}] + 1$ ,  $\alpha \geq 2$  we can estimate

$$\|F''(u)(v, w)\|_{m-\alpha} \leq \|f_{u_{ij}u_{k\ell}} v_{ij} w_{k\ell}\|_{m-\alpha} + \dots$$

$$\leq c(1 + \|u\|_{m-\alpha}) \|v\|_{m-\alpha+2} \|w\|_{m-\alpha+2} + \dots \leq c \|v\|_m \|w\|_m,$$

which shows that  $F''(u)$  is bounded uniformly in  $u$ . The boundness of  $F'(u)$  as well as continuity of  $F'(u)$  and  $F''(u)$  is shown similarly.

Conditions (iia) and (iib) follow from the estimate (6.2) and lemma 6.2, assuming additionally that  $m - \alpha \geq \max([\frac{n}{2}] + 1, 3)$ ,  $\alpha \geq [\frac{n}{2}] + 3$ . Indeed by lemma 2.3

$$a^{m-\alpha} \leq c(\|u\|_{m-\alpha+[\frac{n}{2}]+3} + 1) \leq c, \quad a^2 = o(\delta).$$

To verify (iic) we apply the estimate (6.2) again:

$$(6.8) \quad \|L(u)F(u)\|_{m+8\alpha} \leq c(\|F(u)\|_{m+8\alpha} + p_3\|F(u)\|_{m+8\alpha-1} + \dots + p_{m+8\alpha}\|F(u)\|_2).$$

If we denote  $\tau = \frac{1}{\|u\|_{m+9\alpha}^{\frac{m+9\alpha-[\frac{n}{2}]-5}{m+9\alpha}}}$ , then by lemma 2.4

$$\|u\|_k \leq c\|u\|_{m+9\alpha}^{\frac{k-[\frac{n}{2}]-5}{m+9\alpha-[\frac{n}{2}]-5}} \frac{1}{\|u\|_{[\frac{n}{2}]+5}^{\frac{k-[\frac{n}{2}]-5}{m+9\alpha-[\frac{n}{2}]-5}}} \leq c\tau^{k-[\frac{n}{2}]-5},$$

for  $k = [\frac{n}{2}] + 6, \dots, m + 9\alpha - 1$ . Then as before

$$a^k \leq c(\|u\|_{k+[n/2]+3} + 1) \leq c(\tau^{k-2} + 1), \quad k = 3, \dots, m + 8\alpha;$$

$$p_k \leq c(\tau^{k-2} + 1), \quad k = 3, 4, \dots, m + 8\alpha;$$

$$\|F(u)\|_k \leq c(\tau^{k-[n/2]-3} + 1), \quad k = m - 1, \dots, m + 8\alpha,$$

$$\|F(u)\|_k \leq c, \quad k = 2, \dots, m - 2.$$

Using these estimates in (6.8), we estimate

$$\|L(u)F(u)\|_{m+8\alpha} \leq c(\tau^{m+8\alpha-2} + 1) \leq c(\|u\|_{m+9\alpha} + 1),$$

provided  $m + 8\alpha - 2 \leq m + 9\alpha - [n/2] - 5$ . So that by fixing  $\alpha = [\frac{n}{2}] + 3$ ,  $m = [\frac{n}{2}] + 3 + \max([n/2] + 1, 3)$ ,  $m_0 = m + 8\alpha$ , and  $\delta$  sufficiently small we conclude the proof.

Example. The equation ( $u = u(x)$ ,  $x \in \mathbb{R}^1$ )

$$u = u''^3 + \varepsilon \sin x$$

has a  $2\pi$  periodic solution for  $\varepsilon$  sufficiently small. The equation

$$u = u''^3 + \sin x$$

can be reduced to the one above by scaling  $u$  and  $x$ , and hence it has a solution of period  $2\pi n$ , if  $n$  is a sufficiently large integer.

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