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# Existence and Uniqueness of Solutions for a Class of Non-Autonomous Dirichlet Problems

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**Abstract.** We prove that the semilinear Dirichlet problem for a Laplace equation on a unit ball, involving the nonlinearity $f(r,u) = -a(r)u + b(r)u^p$, with a subcritical $p$, has a unique positive solution, provided $a(r)$ is positive, increasing and convex, while $b(r)$ is positive, decreasing and concave. Moreover, we prove that this solution is non-degenerate. We also present a uniqueness result in case $a(r)$ is negative.

*Key Words and Phrases.* Existence and uniqueness of solutions.

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## 1. Introduction

We study the global curve of positive solutions of a class of Dirichlet problems on balls in $\boldsymbol{R}^n$, with explicit dependence on $r = |x|$

$$\Delta u - \lambda a(r)u + b(r)u^p = 0 \qquad r \in (0,1),\ u = 0 \text{ when } r = 1, \tag{1.1}$$

for subcritical constants $p < (n+2)/(n-2)$ for $n > 2$, and $p < \infty$ for $n = 1, 2$. Here $\lambda$ is a positive parameter. We assume that $a(r)$ is positive, increasing and convex, while $b(r)$ is positive, decreasing and concave. These conditions imply, in particular, that the classical theorem of B. Gidas, W.-M. Ni and L. Nirenberg applies, and hence any positive solution of (1.1) is radially symmetric. We show that all positive solutions of the problem (1.1) lie on a unique smooth curve, which extends without any turns for all $\lambda \geq 0$. In particular, we conclude the existence and uniqueness of positive solutions for any $\lambda > 0$. In case of constant $a(r)$ and $b(r)$ the problem has been studied by a number of people, and is by now completely understood, see e.g. M. K. Kwong [7], P. N. Srikanth [12] and L. Zhang [13]. Allowing the coefficients to depend on $r$ considerably complicates the problem, and so the problems of this type are relatively little studied. We recall that various uniqueness results for the problem (1.1) have been given in the well-known paper of M. K. Kwong and Y. Li [8], and in P. Korman [5].

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In this work we use a global approach. We show that all positive solutions of (1.1) are non-degenerate, i.e. the corresponding linearized problem has only the trivial solution, and hence we can continue for all $\lambda > 0$ the unique positive solution which exists at $\lambda = 0$. To prove non-degeneracy of solutions, we proceed similarly to P. Korman, Y. Li and T. Ouyang [6] and T. Ouyang and J. Shi [10], although the details are more involved now. We obtain a considerable extension of the uniqueness results in M. K. Kwong and Y. Li [8], and in P. Korman [5].

By somewhat different methods we consider the problem

$$\Delta u + \lambda\alpha(|x|)u + u^p = 0 \qquad r \in (0,1),\ u = 0 \text{ when } r = 1, \tag{1.2}$$

where $\alpha(r)$ is positive and decreasing. In Section 5, under some additional conditions we prove uniqueness of positive solutions, giving another extension of the previous results in [8] and [5]. We also consider the case of increasing $\alpha(r)$, and prove existence of a curve of solutions, which admits no turns.

Our results apply to more general classes of equations. We present several generalizations. Throughout the paper we consider only the classical solutions.

## 2. Non-degeneracy of solutions

We study existence and uniqueness of positive classical solution for the problem $(r = |x|)$

$$\Delta u + f(r,u) = 0 \qquad r \in (0,1),\ u = 0 \text{ when } r = 1. \tag{2.1}$$

It follows from B. Gidas, W.-M. Ni and L. Nirenberg [2] that under the condition

$$f_r(r,u) \le 0 \qquad \text{for all } r \in [0,1) \text{ and } u > 0 \tag{2.2}$$

any positive solution of (2.1) is radially symmetric, and hence it satisfies

$$u''(r) + \frac{n-1}{r}u'(r) + f(r,u) = 0 \qquad r \in (0,1),\ u'(0) = u(1) = 0. \tag{2.3}$$

In this section we consider $f(r,u) = -a(r)u + b(r)u^p$, where $1 < p < (n+2)/(n-2)$ for $n \ge 3$, and $1 < p < \infty$ for $n = 1,2$. The functions $a(r), b(r) \in C^2([0,1])$ are assumed to satisfy

$$a(r) > 0, \quad b(r) > 0, \quad a'(r) \ge 0, \quad b'(r) \le 0 \qquad \text{for all } r \in [0,1], \tag{2.4}$$

and

$$a''(r) > 0 \quad \text{and} \quad b''(r) < 0 \qquad \text{for all } r \in [0,1]. \tag{2.5}$$

Notice that these conditions imply (2.2), and so all positive solutions of (2.1) are radially symmetric. I.e. we study the positive solutions of

$$u''(r) + \frac{n-1}{r}u'(r) - a(r)u + b(r)u^p = 0 \qquad r \in (0,1),\ u'(0) = u(1) = 0. \tag{2.6}$$

We shall need the following form of Sturm's comparison theorem, taken from [9]. We consider a differential operator, defined on functions $u = u(r)$ of class $C^2$

$$L[u] \equiv p(r)u'' + q(r)u' + z(r)u,$$

with continuous coefficients $p(r)$, $q(r)$ and $z(r)$.

**Lemma 2.1.** *Assume that on some interval $I \subseteq (-\infty, \infty)$ we have $p(r) > 0$ and*

$$L[u] \geq 0,$$

*while*

$$L[v] \leq 0,$$

*with at least one of the inequalities being strict on a set of positive measure. Then the function $v(r)$ oscillates faster than $u(r)$, provided that they are both non-negative. More precisely, assume that $u(\alpha) = u(\beta) = 0$ for some $\alpha, \beta \in I$, $u(r) > 0$ on $(\alpha, \beta)$, while $v(\alpha) \geq 0$. Then $v(r)$ must vanish on $(\alpha, \beta)$.*

Crucial role will be played by the linearized problem for (2.1)

$$L[w] \equiv w''(r) + \frac{n-1}{r}w'(r) + f_u(r,u)w = 0 \tag{2.7}$$

$$r \in (0,1),\ w'(0) = w(1) = 0.$$

Recall that solution of (2.3) is called non-degenerate if the linearized problem (2.7) admits only the trivial solution. We show next that all positive solutions of (2.6) are non-degenerate.

**Lemma 2.2.** *Under the conditions* (2.4) *and* (2.5) *the problem* (2.7) *has only the trivial solution* $w \equiv 0$.

*Proof.* Let $u(r)$ be a degenerate solution of (2.6), i.e. (2.7) has a non-trivial solution $w(r)$. We shall use a test function $v = ru_r + \mu u$ with a constant $\mu$ to be specified. One easily checks that $v(r)$ satisfies the equation

$$L[v] = \mu(f_u u - f) - 2f - rf_r. \tag{2.8}$$

The sign of the function $L[v]$ is governed by the function $\mu = j(r)$, where

$$j(r) = \frac{2f}{f_u u - f} + \frac{rf_r}{f_u u - f}. \tag{2.9}$$

Indeed, $L[v] > 0$ ($<0$), provided that $j(r) < \mu$ ($>\mu$). (Observe that $f_u u - f > 0$.) For our nonlinearity $f(r,u)$ one easily computes

$$(p-1)j(r) = -2\frac{a(r)}{b(r)}u^{1-p} + 2 - \frac{ra'(r)}{b(r)}u^{1-p} + \frac{rb'(r)}{b(r)}. \tag{2.10}$$

One checks that under our conditions $j'(r) < 0$ for all $r \in (0,1)$, and hence $j(r)$ is a decreasing function. We claim next that

$$f(0,u(0)) > 0. \tag{2.11}$$

Indeed, writing $f(r,u) = u(-a(r) + b(r)u^{p-1})$, we have in the brackets a decreasing function. If inequality (2.11) was violated, we would have $f(r,u(r)) \le 0$ for all $r$. Then multiplying the PDE version of our equation (2.3) by $u$ and integrating over the unit ball, we obtain a contradiction. It follows that

$$j(0) = \frac{2f(0,u(0))}{(p-1)b(0)u^p(0)} > 0.$$

Since $j(r)$ tends to $-\infty$ near $r = 1$, it follows that $j(r)$ changes sign exactly once on $(0,1)$, say at $r = \bar{r}$. Clearly, $j(r) > 0$ for $r \in [0,\bar{r})$, and $j(r) < 0$ for $r \in (\bar{r},1)$.

The sign of the test function $v(r)$ is governed by the function $\mu = h(r) \equiv \frac{-ru_r}{u}$. Indeed, $v(r) > 0$ ($<0$), provided $\mu > h(r)$ ($\mu < h(r)$). We wish to show that $h(r)$ is an increasing function. Compute

$$h' = \frac{(n-2)uu_r + ru_r^2 + ruf(r,u)}{u^2} = \frac{2H}{u^2} + r\frac{uf(u) - 2F(r,u)}{u^2}, \tag{2.12}$$

where we denote $H(r) = (1/2)ru_r^2 + (1/2)(n-2)uu_r + rF(r,u)$, and $F(r,u) = \int_0^u f(r,t)dt$. The idea of introducing the function $H(r)$ in a similar context is due to T. Ouyang and J. Shi [10]. One verifies that $H(r)$ satisfies

$$H' + \frac{n-1}{r}H = nF - \frac{n-2}{2}uf(u) + rF_r(r,u) \equiv G(r) \qquad H(0) = 0. \tag{2.13}$$

Observe that $G(r) = -A(r)u^2 + B(r)u^{p+1}$, where

$$A(r) = a(r) + \frac{1}{2}ra'(r) > 0, \tag{2.14}$$

$$B(r) = b(r)\left(\frac{n}{p+1} - \frac{n-2}{2}\right) + r\frac{b'(r)}{p+1}.$$

Our conditions imply that $A(r)$ is positive and increasing, while $B(r)$ is decreasing on $(0,1)$. (Observe that $n/(p+1)-(n-2)/2>0$ for subcritical $p$.)

Near $r=1$ the function $G(r)$ is negative, since its first term dominates. We claim that either $G(r)<0$ for all $r\in(0,1)$ or $G(r)$ changes its sign exactly once on $(0,1)$, with $G(0)>0$. Since $B(r)$ is decreasing, and $B(0)>0$, it follows that either $B(r)>0$ for all $r\in(0,1)$, or there is an $\tilde{r}\in(0,1)$, so that $B(r)>0$ on $[0,\tilde{r})$, and $B(r)<0$ on $(\tilde{r},1]$. Let us assume first that $B(r)>0$ for all $r$. Setting $G(r)$ to zero, we rewrite it in the form

$$u^{p-1}=\frac{A(r)}{B(r)}. \tag{2.15}$$

On the left in (2.15) we have a decreasing function, and an increasing one on the right, and also $0=u^{p-1}(1)<A(1)/B(1)$. Hence, either $u^{p-1}(r)<A(r)/B(r)$ for all $r\in(0,1)$, which implies that $G(r)<0$ for all $r\in(0,1)$, or else (2.15) has exactly one solution, i.e. $G(r)$ changes its sign exactly once on $(0,1)$. Turning to the second case, when $B(r)$ changes sign at $\tilde{r}$, we observe that $G(r)$ is negative on $[\tilde{r},1)$, while on $(0,\tilde{r})$ the equation (2.15) has either one solution, or no solution, so that we can proceed as before to prove the claim.

Next we show that

$$H(r)>0 \qquad \text{for all } r\in(0,1]. \tag{2.16}$$

Observe that $H(1)=(1/2)u'^2(1)>0$. Consider first the case when $G(r)<0$ for all $r\in(0,1]$. We multiply (2.13) by $r^{n-1}$ and integrate

$$H(1)-r^{n-1}H(r)=\int_r^1 r^{n-1}G(r)dr<0,$$

and (2.16) follows. The other possibility is that $G(r)>0$ on $[0,\hat{r})$, and $G(r)<0$ on $(\hat{r},1]$, for some $\hat{r}\in(0,1)$. Then $H(r)>0$ on $(\hat{r},1]$, as before. If $H(r)$ were to become negative somewhere on $[0,\hat{r})$, it would have to achieve negative minimum at some $r_0\in(0,\hat{r})$, since $H(0)=0$. At $r_0$ we have a contradiction in (2.13), since the quantity on the left is negative, and the one on the right is positive.

Since $uf(u)-2F(u)=(1-2/(p+1))bu^{p+1}>0$, it follows from (2.12) that $h(r)$ is an increasing function. Recall that $j(r)$ was a decreasing function. Since $0=h(0)<j(0)$, and the order is reversed near $r=1$ (where $h(r)$ tends to infinity), it follows that $h(r)$ and $j(r)$ intersect exactly once, at some $r_1$. Let $\mu_1=h(r_1)=j(r_1)$, and we now fix our test function $v(r)=ru_r+\mu_1 u$.

On $(0,r_1)$ we have $v>0$, while $Lv<0$. On $(r_1,1)$ it is $v<0$, while $Lv>0$. By Lemma 2.2 we see that $w(r)$ is of one sign, since it cannot vanish on either $(0,r_1]$ or $[r_1,1)$. Since $uf_u-f>0$, it is well-known that $w(r)$ cannot remain positive (or just use a test function $v = u$ and Lemma 2.2). We conclude that $w$ is identically zero. $\diamondsuit$

We shall also need a limiting case of the preceeding lemma.

**Lemma 2.3.** *Assume that* $a(r) = 0$ *for all* $r \in [0, 1)$, *while* $b(r)$ *satisfies the conditions* (2.4) *and* (2.5). *Then the problem* (2.7) *admits only the trivial solution* $w \equiv 0$.

*Proof.* The proof is similar to that of Lemma 2.2 above, so that we only sketch the argument. We use the same test function $v(r) = ru_r + \mu u$, and define $j(r)$ and $h(r)$ the same way. This time $j(r) = (1/(p-1))(2 + rb'/b)$, which is a decreasing function by our assumptions. In (2.12) $uf(u) - 2F(u)$ is the same expression as before, and hence positive. This time $G(r) = B(r)u^{p+1}$, with the same $B(r)$ as in (2.14). We see that $B(0) > 0$, while $B(r)$ is a decreasing function. So that there are two cases to consider.

*Case* (i). $B(r) > 0$ for all $r \in (0, 1)$. Then from (2.13)

$$r^{n-1}H(r) = \int_0^r \xi^{n-1}B(\xi)u^{p+1}(\xi)d\xi > 0,$$

i.e. $H(r) > 0$ for all $r \in (0, 1)$.

*Case* (ii). $B(r)$, and hence $G(r)$ changes sign exactly once on $(0, 1)$. Then we prove that $H(r) > 0$ for all $r \in (0, 1)$ exactly as before.

We conclude from (2.12) that $h(r)$ is an increasing function, and complete the proof exactly as before. $\diamondsuit$

## 3. Existence and uniqueness of solutions

We can now prove our main result.

**Theorem 3.1.** *Assume the conditions* (2.4) *and* (2.5) *hold. Then for any* $\lambda \geq 0$ *the problem* (1.1) *has a unique positive solution. Moreover, all positive solutions of* (1.1) *are non-degenerate, of Morse index equal to one, and they lie on a unique smooth curve of solutions.*

*Proof.* We begin by proving the theorem in the case $\lambda = 0$. We claim that the problem

$$\Delta u + b(r)u^p = 0, \qquad r \in (0, 1),\ u = 0 \text{ for } r = 1 \tag{3.1}$$

has a unique positive solution. In case $b(r)$ is a constant, existence and uniqueness of positive solution is well-known, see e.g. [4]. Moreover, the Morse index of the solution is equal to one. This is known in general for the solutions of the mountain-pass type, see e.g. K. C. Chang [1], or a simple proof for this particular case can be found in [4]. We now embed (3.1) into a family of problems

$$(3.2) \qquad \Delta u + [\nu b(r) + (1-\nu)b(1)]u^p = 0, \qquad r \in (0,1),\ u = 0 \text{ for } r = 1,$$

depending on a parameter $\nu$, $0 \le \nu \le 1$. As we vary $\nu$, we can apply the implicit function theorem to continue the solution, since by Lemma 2.3 all solutions of (3.2) are non-degenerate. By the a priori estimates of B. Gidas and J. Spruck [3] the solutions stay bounded, and hence they can be continued for all $\nu$. At $\nu = 1$ we conclude existence and uniqueness of positive solutions for our problem (1.1) at $\lambda = 0$. (If there were more than one solution at $\nu = 1$, we would have more than one solution at $\nu = 0$, a contradiction.) Moreover, the Morse index of this solution is one, since eigenvalues of the linearized problem for (1.1) change continuously, and they cannot cross zero, since all solutions are non-degenerate (and so the number of negative eigenvalues at $\nu = 0$, is the same as at $\nu = 1$, i.e. one).

The proof of the theorem for any $\lambda > 0$ follows essentially the same argument. By Lemma 2.2 and the a priori estimates of B. Gidas and J. Spruck [3], we can continue our solution for all $\lambda$ (starting at $\lambda = 0$). If there were more than one solution at some $\lambda > 0$, we could continue it back for decreasing $\lambda$, and obtain more than one solution at $\lambda = 0$, contradicting the uniqueness at $\lambda = 0$, that we have just proved. $\diamond$

## 4. A case of positive coefficient in the linear term

In this section we consider positive radial solutions of

$$(4.1) \qquad \Delta u + \lambda\alpha(|x|)u + u^p = 0 \qquad r \in (0,1),\ u = 0 \text{ when } r = 1,$$

i.e. we shall again be looking at the problem (2.3) and the corresponding linearized problem (2.7), this time with $f(r,u) = \alpha(r)u + u^p$, where the given function $\alpha(r)$ is continuously differentiable and positive. After some preliminary results (which we present in more generality than is needed for this section), we prove existence of a curve of non-degenerate solutions.

The following two formulas are straightforward modifications of the corresponding results for the autonomous case from [6] and [10].

**Lemma 4.1.** *Let $u(r)$ be a degenerate positive solution of* (2.3)*, and $w(r)$ the corresponding solution of the linearized problem* (2.7). *Then*

$$(4.2) \qquad \int_0^1 f(r,u)wr^{n-1}\,dr + \frac{1}{2}\int_0^1 f_r(r,u)wr^n\,dr = \frac{1}{2}u'(1)w'(1),$$

$$(4.3) \qquad \int_0^1 f(r,u)wr^{n-1}\,dr = \int_0^1 uf_u(r,u)ur^{n-1}\,dr.$$

*Proof.* The function $v(r) = ru'(r) - u'(1)$ satisfies

$$v'' + \frac{n-1}{r}v' + f_u(r,u)v = -2f(u) - f_u(r,u)u'(1) - rf_r(r,u) \tag{4.4}$$

$$\text{if } r < 1,\ v = 0 \text{ for } r = 1.$$

Combining (4.4) with the linearized problem

$$w'' + \frac{n-1}{r}w' + f_u(r,u)w = 0 \qquad \text{if } r < 1,\ w = 0 \text{ for } r = 1, \tag{4.5}$$

we have

$$\int_0^1 f(r,u)wr^{n-1}\,dr + \frac{1}{2}\int_0^1 f_r(r,u)wr^n\,dr = -\frac{1}{2}u'(1)\int_0^1 f_u(r,u)wr^{n-1}\,dr. \tag{4.6}$$

Integrating the equation in (4.5)

$$\int_0^1 f_u(r,u)wr^{n-1}\,dr = -w'(1). \tag{4.7}$$

Combining (4.6) and (4.7), we have (4.2).

Turning to the second part, we observe that $u(r)$ satisfies

$$u'' + \frac{n-1}{r}u' + f_u(r,u)u = f_u(r,u)u - f(r,u), \tag{4.8}$$

$$\text{if } r < 1,\ u = 0 \text{ for } r = 1.$$

Combining (4.8) with (4.6), we conclude (4.3). $\diamondsuit$

Recall that we have defined $F(r,u) = \int_0^u f(r,t)dt$.

**Lemma 4.2.** *Any solution of* (2.3) *satisfies*

$$\int_0^1 \left(nF - \frac{n-2}{2}uf(u)\right)r^{n-1}\,dr + \int_0^1 F_r(r,u)r^n\,dr = \frac{1}{2}u'^2(1). \tag{4.9}$$

*Proof.* Just integrate (2.13). $\diamondsuit$

The following is the crucial lemma. It is based on J. Shi [11], see also P. N. Srikanth [12]. We present it in a more general form than is necessary here. Notice that we allow two sets of conditions, with the second one given in brackets.

**Lemma 4.3.** *Let $u(r)$ be a degenerate positive solution of* (2.3), *and $w(r)$ the corresponding solution of the linearized problem* (2.7). *Assume that for some $\gamma > 1$, and for all $r \in [0,1)$ and $u > 0$, the following inequalities hold*

$$f_r(r,u) \le 0 \ \ (\ge 0), \tag{4.10}$$

$$uf_u(r,u) - f(r,u) > 0, \tag{4.11}$$

$$uf_u(r,u) - \gamma f(r,u) > 0 \ \ (<0), \tag{4.12}$$

$$A \equiv (\gamma - 1)\left[nF(r,u) - \frac{n-2}{2}uf(r,u)\right] + [uf_u(r,u) - \gamma f(r,u)]u < 0 \ \ (>0), \tag{4.13}$$

$$\frac{1}{2}uf_r(r,u) - F_r(r,u) \ge 0 \ \ (\le 0). \tag{4.14}$$

*Then $w(r)$ cannot vanish exactly once inside* $(0,1)$.

*Proof.* Assume on the contrary that $w(r)$ vanishes exactly once at some $\xi \in (0,1)$, i.e. we may assume that $w < 0$ on $[0,\xi)$, and $w > 0$ on $(\xi,1)$. By scaling $w$, we may achieve

$$w(r) < u(r) \qquad \text{for all } r \in [0,1). \tag{4.15}$$

Now, if we scale $w(r)$ up, i.e. consider $\beta w$, with $\beta > 1$, we will have the inequality (4.15) violated, provided we choose $\beta$ large enough. Let $\beta_0$ be supremum of $\beta$'s for which the inequality (4.15) holds, and let us consider the corresponding solution of the linearized equation $\beta_0 w(r)$, which we shall still denote by $w(r)$. There are two possibilities to consider.

*Case* (i). There is an $\eta \in (\xi,1)$ so that $w(\eta) = u(\eta)$, while $w(r) \le u(r)$ for all $r \in [0,1)$. We use a test function $v(r) = w(r) - u(r)$. With $L[v]$ as defined by (2.7), we have

$$L[v] = -L[u] = -(uf_u(r,u) - f(r,u)) < 0 \tag{4.16}$$

by our assumption (4.11). On the interval $(\xi,1)$ we have $v \le 0$, $L[v] < 0$, and $v(\xi) = v(1) = 0$. It follows by Lemma 2.1 that $v$ oscillates slower than $w$ on this interval, and hence $w(r)$ must vanish on $(\xi,1)$, a contradiction.

*Case* (ii). $w'(1) = u'(1)$, while $w(r) < u(r)$ for all $r \in [0,1)$. Then the right hand sides in (4.2) and (4.9) are equal. Subtracting these equations,

$$\int_0^1 fwr^{n-1}\,dr + \int_0^1 \left(\frac{1}{2}f_r w - F_r\right) r^n\,dr - \int_0^1 \left(nF - \frac{n-2}{2}uf\right) r^{n-1}\,dr = 0. \tag{4.17}$$

Using (4.3), we rewrite this as

$$\int_0^1 uf_u w r^{n-1}\,dr + \int_0^1 \left(\frac{1}{2} f_r w - F_r\right) r^n\,dr - \int_0^1 \left(nF - \frac{n-2}{2} uf\right) r^{n-1}\,dr = 0. \tag{4.18}$$

From (4.18) we subtract (4.17) times $\gamma > 1$,

$$\begin{aligned} &\int_0^1 (uf_u - \gamma f) w r^{n-1}\,dr + (\gamma - 1)\int_0^1 \left(nF - \frac{n-2}{2} uf\right) r^{n-1}\,dr \\ &\qquad + (1-\gamma)\int_0^1 \left(\frac{1}{2} f_r w - F_r\right) r^n\,dr = 0. \end{aligned} \tag{4.19}$$

Assume for definiteness that the first set of conditions hold. Using our conditions (4.10) and (4.11), and the inequality (4.15), we conclude from (4.19)

$$\int_0^1 A r^{n-1}\,dr + (1-\gamma)\int_0^1 \left(\frac{1}{2} f_r u - F_r\right) r^n\,dr > 0, \tag{4.20}$$

which is a contradiction, since by our assumptions (4.13) and (4.14) the left hand side is non-positive. $\diamondsuit$

**Theorem 4.1.** *Consider the problem* (4.1)*, with* $1 < p < (n+2)/(n-2)$*, and where* $\alpha(r) \in C^1[0,1) \cap C[0,1]$ *satisfies*

$$\alpha(r) > 0, \qquad \alpha'(r) \geq 0, \qquad \text{for } r \in [0,1). \tag{4.21}$$

*Then the problem* (4.1) *has a smooth curve of positive radial solutions for all* $\lambda \geq 0$*. This curve admits no turns, and the Morse index of any solution on this curve is one.*

*Proof.* As we mentioned previously, at $\lambda = 0$ the problem has a positive solution of Morse index one. We now continue this solution for $\lambda > 0$ using the implicit function theorem in the space of radially symmetric functions of class $C^2$, until we reach a degenerate solution, at which the linearized problem (2.7) has a non-trivial solution $w(r)$. Since eigenvalues change continuously, $\mu = 0$ is either first or second eigenvalue of the linearized eigenvalue problem

$$w'' + \frac{n-1}{r} w' + \lambda\alpha(r) w + p u^{p-1} w + \mu w = 0 \qquad r \in (0,1),\ w = 0 \text{ when } r = 1,$$

and $w(r)$ is the corresponding eigenfunction. One checks that the Lemma 4.3 applies (the second set, with $\gamma = p$), and hence $w(r)$ cannot vanish exactly once, and hence it cannot be the second eigenfunction. Since $uf_u - f > 0$ it is well known that $w(r)$ cannot be positive either. Hence degenerate solution is never encountered, and we can continue the solution curve for all $\lambda > 0$ as in the Theorem 3.1. $\diamondsuit$

The theorem does not allow us to conclude uniqueness of solutions, but we can say that any other solution (not on the curve just described) would have to be of Morse index two or higher. Non-radial positive solutions are also possible in this case. In the next section we present an uniqueness result for (4.1), under a different set of conditions.

## 5. Positivity for the linearized problem

We present a general result on the positivity of solutions for the linearized problem 2.7, and then apply it to prove uniqueness for the problem (4.1).

**Lemma 5.1.** *Let $u(r)$ be a degenerate positive solution of* (2.3), *and assume that the following conditions are satisfied for all $r \in [0,1)$ and $u \geq 0$*

$$f(r,0) = 0, \quad f_r(r,u) \leq 0, \quad f_{rr}(r,u) \leq 0, \quad f_{ru}(r,u) \leq 0, \quad f_u(r,u) > 0, \tag{5.1}$$

$$B \equiv f_r f_u - f f_{ru} \leq 0, \tag{5.2}$$

$$C \equiv 2f_u^2 - nff_{uu} + r(f_{ru}f_u - f_{uu}f_r) > 0. \tag{5.3}$$

*Then any non-trivial solution of the linearized problem* (2.7) *is of one sign.*

*Proof.* This time we will use the test function $v = ru' + (n-2)u + \alpha$, with a constant $\alpha$ to be specified. Compute

$$L[v] = (n-2)uf_u - nf - rf_r + \alpha f_u \equiv g_\alpha(u). \tag{5.4}$$

The sign of the test function $v(r)$ is governed by the function $\alpha = h(r) \equiv -ru' - (n-2)u$. Indeed, $v > 0$ $(<0)$ when $h(r) < \alpha$ $(>\alpha)$. Similarly, the sign of $g_\alpha(u)$ is governed by $\alpha = j(r) \equiv \frac{-(n-2)uf_u + nf + rf_r}{f_u}$. Indeed, $L[v] > 0$ $(<0)$ when $j(r) < \alpha$ $(>\alpha)$. Write $j(r) = j_1(r) + j_2(r)$, where we denote $j_1(r) = \frac{rf_r}{f_u}$ and $j_2(r) = \frac{-(n-2)uf_u + nf}{f_u}$. In view of our conditions

$$\begin{aligned} j_1'(r) &= \frac{(f_r + rf_{rr} + rf_{ru}u')f_u - rf_r(f_{ru} + f_{uu}u')}{f_u^2} \\ &< ru' \frac{f_{ru}f_u - f_{uu}f_r}{f_u^2}, \end{aligned} \tag{5.5}$$

for all $r \in (0,1)$. We also have

$$j_2'(r) = \frac{(2f_u^2 - nff_{uu})u' + nB}{f_u^2} < \frac{2f_u^2 - nff_{uu}}{f_u^2} u'. \tag{5.6}$$

Combining (5.5) and (5.6), we see that

$$j'(r) < \frac{C}{f_u^2} u' < 0,$$

for all $r \in (0,1)$, and so $j(r)$ is a decreasing function. Since $h'(r) = rf > 0$ the function $h(r)$ is increasing for all $r \in (0,1)$. (Observe that our conditions imply that $f(r,u) > 0$.) We now consider two cases.

*Case* (i). $j(0) > h(0)$. Since $j(1) < 0$, while $h(1) > 0$, it follows that the functions $h(r)$ and $j(r)$ have a unique point of intersection, call it $\bar{r}$. We now fix $\alpha = h(\bar{r})$. Then on $(0,\bar{r})$ we have $v > 0$ and $L[v] < 0$, while the opposite inequalities hold on $(\bar{r},1)$. It follows that $w(r)$ cannot vanish.

*Case* (ii). $j(0) \le h(0)$. This time we fix $\alpha = h(0)$. We have $v < 0$ and $L[v] > 0$ on the entire interval $(0,1)$. Again, $w(r)$ cannot vanish. ◇

We now return to the problem (4.1).

**Theorem 5.1.** *Assume that* $\alpha(r) \in C^2[0,1)$ *is a positive function, and the following conditions hold*

$$
1 < p \le \frac{n}{n-2}, \quad \textit{for } n \ge 3, \qquad 1 < p < \infty \quad \textit{for } n = 1,2, \tag{5.7}
$$

$$
a'(r) \le 0, \qquad \textit{and} \qquad a''(r) \le 0 \qquad \textit{for all } r \in (0,1),
$$

$$
(r^2\alpha(r))' > 0, \qquad \textit{for all } r \in (0,1),
$$

$$
[4 - n(p-1)]\alpha(r) + (2-p)r\alpha'(r) > 0.
$$

*Then for any* $\lambda \ge 0$ *the problem* (4.1) *has a unique positive solution. Moreover, all solutions of* (4.1) *are non-degenerate of Morse index equal to one, and they lie on a unique smooth curve of solutions.*

*Proof.* One checks that the Lemma 5.1 applies here, implying that any non-trivial solution of the linearized problem is of one sign. Since here $uf_u - f > 0$, it follows that any solution of the linearized problem (2.7) must vanish on $(0,1)$, and hence (2.7) can have only the trivial solution, i.e. any solution of (4.1) is non-degenerate. Similarly to the Theorem 4.1, we obtain a curve of positive solutions. This curve of solutions exhausts the solution set. Indeed, any other solution could be continued for decreasing $\lambda$, contradicting the uniquenes of solution at $\lambda = 0$. ◇

*Remarks.*

1. It is straightforward to give a similar uniqueness result in case $f(r,u) = \alpha(r)u^q + \beta(r)u^p$, with $1 < q < p < n/(n-2)$, and $\beta'(r) \le 0$.
2. M. K. Kwong and Y. Li [8] have proved uniqueness, assuming that $1 < p \le (n+2)/(n-2)$ and the function $r^\beta\alpha(r)$ is non-decreasing, where $\beta = 2(n-1)(p-1)/(p+3)$. Clearly, we have several extra restrictions. But since $\beta < 2$ for $n \ge 4$, our third condition is less

restrictive for differentiable $\alpha(r)$, thus providing an extension for all $n \geq 4$. Notice that our result provides some extra information, and it appears easier to extend to other equations, as in the preceeding remark.

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