



An accurate computation of the global solution curve for the Gelfand problem through a two point approximation

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Abstract

When one solves numerically the Gelfand boundary value problem in two dimensions (with $u = u(x, y)$, $r = \sqrt{x^2 + y^2}$ and λ a positive parameter)

$$\Delta u + \lambda e^u = 0 \quad \text{for } r < 1, \quad u = 0 \quad \text{when } r = R.$$

with 10 mesh points, two spurious turns appear on the solution curve. When one increases the number of mesh points, the spurious turns persist, occurring at smaller values of λ . It turns out that spurious turns are avoided in the other direction, by using two to six mesh points. We prove this in case of two mesh points. This gives us the correct form of the solution curve, and the accuracy can then be improved by using a linear search on $u(0)$, combined with shooting.

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1. Introduction

One common way to numerically approximate solutions of a nonlinear boundary value problem is to use finite differences approximation. The resulting system of nonlinear algebraic equations can be solved by e.g. Newton's method, providing an approximation of solution at mesh points. It is then natural to expect that solution of the difference equation approaches the

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desired solution of the differential equation, as one decreases the mesh size. However it has been known for some time that the difference approximations tend to have more solutions than the differential equations that they came from, see e.g. [1] or [11] for some of the early references, or a recent paper of McGough [8] (which studies the Gelfand problem in three dimensions). In particular, some solutions of the difference approximations do not correspond to any solutions of the corresponding differential equations. These are so-called *spurious* solutions. Decreasing the step size will eliminate the spurious solutions at a given value of the parameter, but they may persist at some other values of the parameter, preventing an accurate computation of the entire solution curve. We will be mostly concerned with solving the Gelfand boundary value problem in two dimensions (with $u = u(x, y)$, $r = \sqrt{x^2 + y^2}$)

$$\Delta u + \lambda e^u = 0 \quad \text{for } r < 1, \quad u = 0 \quad \text{when } r = R. \quad (1)$$

This much studied problem comes up in the theory of combustion, see e.g. [2]. Here λ is a positive parameter. Observe that by maximum principle any solution of (1) is positive, and then by the theorem of Gidas et al. [7] any solution is radially symmetric, i.e. $u = u(r)$, and it satisfies

$$u'' + \frac{1}{r}u' + \lambda e^u = 0, \quad \text{when } r \in (0, R), \quad u'(0) = u(R) = 0. \quad (2)$$

Moreover, it follows from [7] that $u'(r) < 0$ for all $r \in (0, R]$. It is also known that the maximum value $u(0)$ uniquely determines the solution $(\lambda, u(r))$ of (2), see e.g. [3]. We can therefore represent the solutions of (2) as curves in $(\lambda, u(0))$ plane. The problem (2) has been completely analyzed for all space dimensions by Joseph and Lundgren [5] (see also [2] for a nice exposition). It turns out that all solutions lie on a unique curve in $(\lambda, u(0))$ plane. In two dimensions this curve starts at $(0, 0)$ then bends back at $\lambda_0 = 2$, and then continues without any turns with $u(0) \rightarrow \infty$ as $\lambda \rightarrow 0$. When we solved the problem numerically for $R = 1$ using 10 mesh points (see below the details of our implementation), the solution curve indeed made a turn at $\lambda_0 = 2$, but then it made two extra turns at $\lambda_1 = 0.19$ and $\lambda_2 = 0.229$, see Fig. 1. When we increased the number of mesh points, the solution curve still had two extra turns, with the corresponding values of λ_1 and λ_2 getting smaller. It turns out that the way to get rid of the spurious turns is in the opposite direction: decreasing the number of mesh points. We prove that for two mesh points no spurious turns occur. Our numerical experiments show that there are still no spurious turns for ≤ 6 mesh points, see Fig. 2 where solution of (2) is computed with five mesh points. This leads to the following strategy for solving the problem (2). Solve the problem (2) with a small number of mesh points. Then there are no spurious turns, and we obtain reasonably accurate values of $u(0)$ for all $\lambda > 0$. It is then easy to improve the values of $u(0)$, by using linear search on $u(0)$, coupled with the computation of the entire solution. To obtain the solution, one needs to solve

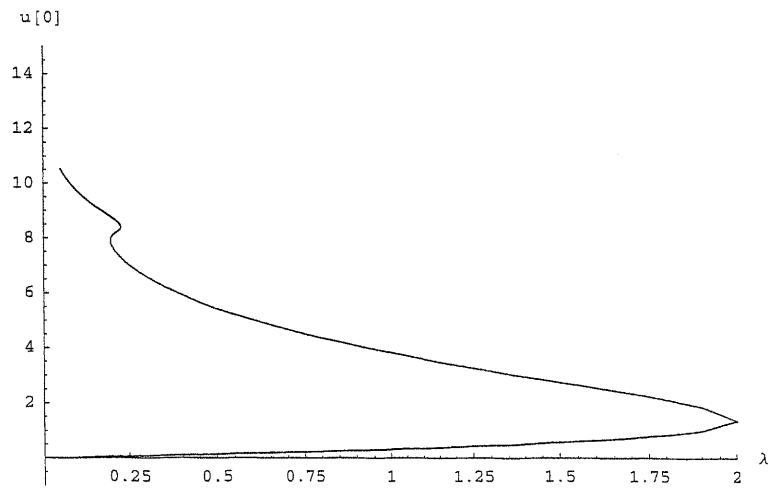


Fig. 1. Two spurious turns.

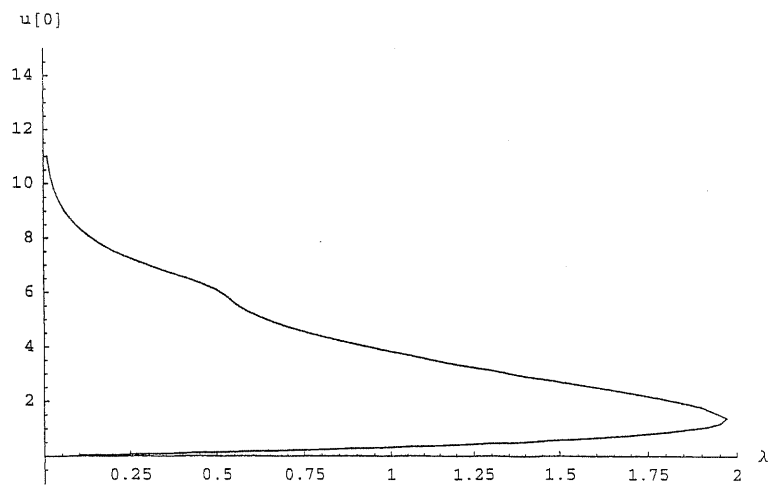


Fig. 2. Solution curve with five mesh points.

an initial value problem, starting at $r = 0$, for which standard (fast and accurate) algorithms are available, e.g. Runge–Kutta.

In Section 3, following Govaerts [4], we show that for the one-dimensional version of the problem (1) spurious symmetry breaking occurs even in case of two mesh points.

We wish to mention that Mickens had studied extensively the ways to avoid spurious solutions by using nonstandard finite difference schemes, see e.g. [9].

2. Numerical implementation and the two-point scheme

We now describe the numerical approximation of (2). We divide the interval $[0, R]$ into n equal pieces of length $h = R/n$ each, with $r_i = ih$, $i = 0, 1, \dots, n$. We shall denote by u_i the numerical approximation of $u(r_i)$. To approximate the Neumann condition at $r = 0$ it is customary to introduce a fictitious mesh point $r_{-1} = -h$, with u_{-1} the fictitious value at r_{-1} , and then write the equation (2) at the boundary point $r = 0$, in order to exclude r_{-1} , see e.g. [12]. Letting $r \rightarrow 0$, and using that $\lim_{r \rightarrow 0} u'(r)/r = u''(0)$, we conclude that

$$2u''(0) + \lambda e^{u(0)} = 0. \quad (3)$$

Using the standard central difference approximation, we replace (3) by

$$2 \frac{u_1 - 2u_0 + u_{-1}}{h^2} + \lambda e^{u_0} = 0. \quad (4)$$

Since $u'(0) = 0$, we conclude that $u_{-1} = u_1$, and hence (4) takes the form

$$4u_1 - 4u_0 + \lambda h^2 e^{u_0} = 0. \quad (5)$$

For other mesh points x_i , with $i = 1, 2, \dots, n-1$, we use the standard central difference approximation

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + \frac{1}{ih} \frac{u_{i+1} - u_{i-1}}{2h} + \lambda e^{u_i} = 0. \quad (6)$$

This is a system of n algebraic equations for $(u_0, u_1, \dots, u_{n-1})$, which we solved as a function of $\lambda > 0$ by the predictor–corrector method, using the Newton's method with 4–6 steps as a corrector. As we mentioned above, this implementation results in two spurious turns of the solution curve for all $n \geq 10$ that we tried.

We now completely analyze the $n = 2$ case. The second equation is then (here $i = 1$, and $u_2 = 0$)

$$\frac{1}{2}u_0 - 2u_1 + \lambda h^2 e^{u_1} = 0. \quad (7)$$

Setting $\mu = \lambda h^2 > 0$, we solve (5) and (7) as a 2×2 system

$$\begin{aligned} -4u_0 + 4u_1 + \mu e^{u_0} &= 0 \\ \frac{1}{2}u_0 - 2u_1 + \mu e^{u_1} &= 0. \end{aligned} \quad (8)$$

We are looking for positive solution of (8), $(u_0, u_1) > 0$.

Theorem 1. *For any positive $\mu > 0$ the problem (8) has at most two positive solutions.*

Proof. Solving for u_1 from the first equation in (8), and using that in the second one, we obtain

$$\frac{3}{2}u_0 - \frac{\mu}{2}e^{u_0} - \mu e^{u_0 - \frac{\mu}{4}e^{u_0}} = 0.$$

Set $g(x) = (3/2)x - (\mu/2)e^x - \mu e^{x - \frac{\mu}{4}e^x}$. We will show that $g''(x) < 0$ for all $x > 0$, and the proof will follow, since a concave function can have at most two roots. Compute

$$\begin{aligned} g'(x) &= \frac{3}{2} - \frac{\mu}{2}e^x - \mu e^{x - \frac{\mu}{4}e^x} \left(1 - \frac{\mu}{4}e^x\right), \\ g''(x) &= -\frac{\mu}{2}e^x - \mu e^{x - \frac{\mu}{4}e^x} \left[\left(1 - \frac{\mu}{4}e^x\right)^2 - \frac{\mu}{4}e^x \right] \\ &= -\mu e^x \left[\frac{1}{2} + e^{-\frac{\mu}{4}e^x} \left(1 - \frac{3}{4}\mu e^x + \frac{\mu^2}{16}e^{2x}\right) \right]. \end{aligned}$$

Setting $t = \mu e^x > 0$, we see that the quantity in the square bracket

$$\phi(t) \equiv \frac{1}{2} + e^{-t/4} \left(1 - \frac{3}{4}t + \frac{1}{16}t^2\right) > 0 \quad \text{for all } t > 0.$$

Indeed, the critical points of $\phi(t)$ turn out to be at $t = 4$ and $t = 16$, with $t = 4$ being the point of absolute minimum on $[0, \infty)$, and the minimum value is $\phi(4) = \frac{1}{2} - e^{-1} > 0$. \square

Remark. We used in an essential way the particular form of our nonlinearity $f(u) = e^u$. We expect a similar result to hold for many convex $f(u)$, but cannot prove that.

3. Spurious symmetry breaking in one dimension

We now consider the problem (1) in one space dimension. We assume that $u = u(x)$ is a positive solution of

$$u'' + \lambda e^u = 0 \quad \text{for } x \in (0, 1), \quad u(0) = u(1) = 0. \quad (9)$$

As mentioned above, the positive solutions of (9) are symmetric about $x = 1/2$, the mid-point of the interval. However, even with two mesh points spurious symmetry breaking occurs. This result is due to Govaerts [4]. We present here a different exposition, which is more in line with the bifurcation theory analysis, that was used extensively in recent years by Li, Ouyang, Shi and the present author, see e.g. [6,10], and which appears to be a little easier. Crucial to the bifurcation theory analysis is consideration of the linearized problem for (9)

$$w'' + \lambda e^u w = 0, \quad \text{for } x \in (0, 1), \quad w(0) = w(1) = 0. \quad (10)$$

Letting $h = 1/3$, we introduce two interior mesh points $x_1 = 1/3$, $x_2 = 2/3$ and denote by u_1 , u_2 the numerical approximations of $u(x_1)$ and $u(x_2)$ respectively. Then the problem (9) is approximated by the algebraic system

$$\begin{aligned} -2u_1 + u_2 + h^2 \lambda e^{u_1} &= 0, \\ u_1 - 2u_2 + h^2 \lambda e^{u_2} &= 0. \end{aligned} \quad (11)$$

Since by the symmetry $u_1 = u_2 \equiv u$, (11) is equivalent to a single equation

$$u = h^2 \lambda e^u \equiv \mu. \quad (12)$$

Clearly there exists a critical $\lambda = \lambda_0$ so that the Eq. (12) has either two, one or no solution when $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$ respectively. This mirrors precisely the solution set of the original problem (9). We now discretize the linearized equation (10). Using that $h^2 \lambda e^{u_1} = h^2 \lambda e^{u_2} = h^2 \lambda e^u = \mu$, we have

$$\begin{aligned} -2w_1 + w_2 + \mu w_1 &= 0, \\ w_1 - 2w_2 + \mu w_2 &= 0. \end{aligned} \quad (13)$$

The linear system (13) will have nontrivial solutions, provided its determinant is zero. This happens at $\mu = 1$, i.e. when $(u = 1, \lambda = (1/h^2)e^{-1} = 9e^{-1})$, and at $\mu = 3$, which corresponds to $(u = 3, \lambda = (3/h^2)e^{-3} = 27e^{-3})$. At $\mu = 1$ the solution set of (13) is spanned by $(w_1 = 1, w_2 = 1)$, while at $\mu = 3$ the solution set of (13) is spanned by $(w_1 = 1, w_2 = -1)$. It is then not hard to show that at $(u = 1, \lambda = 9e^{-1})$ a simple turn occurs, while further along the solution curve at $(u = 3, \lambda = 27e^{-3})$ there is a symmetry breaking bifurcation, see [4].

Remark. Symmetry breaking numerical solutions are of course spurious solutions, since all solutions of the original problem (9) are symmetric. However, these spurious solutions do not represent a serious computational challenge (unlike the spurious turns from the previous section), since a good path following algorithm will stay on the curve of symmetric solutions. Moreover, symmetry can be “forced” for the numerical solution. For example, in case of two mesh points we can replace the system (11) by a single equation (12).

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