Analysis of the Convergence Behavior of the Static Localized Nonlinear Approximation for an Electromagnetic Scattering Problem

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Abstract

A study of the convergence behavior of the static localized nonlinear approximation introduced by Habashy, Groom, and Spies [3] is furnished for a specific case of the time-harmonic Maxwell equations. Some asymptotics are used to explain the robustness of this scheme in the high conductivity case.

1. Introduction:

We will study the static localized nonlinear(SLN) approximation introduced in [3] applied to the nondimensionalized time-harmonic Maxwell equations,

$$\nabla \times E - i\gamma H = -M_s \quad \text{and} \quad \nabla \times H - \gamma\sigma E = 0 \quad \text{in} \quad \mathbf{R}^3 \tag{1}$$

where

$$\sigma(x) = \begin{cases} \sigma_s/\sigma_b & \text{if } x \in \Omega \\ 1 & \text{if } x \in \mathbf{R}^3 - \Omega. \end{cases}$$

The set Ω is the nondimensionalized domain which represents the scatterer and γ is a parameter which we will assume satisfies $0 < \gamma < 1$. The positive constants σ_s and σ_b are conductivities in the scatterer and background, respectively. The functions E and H are the rescaled electric and magnetic fields, respectively, and M_s is the magnetic source.

As in Habashy et al [3] we will focus on the case where the scatterer Ω is a simple closed region, there is a source (transmitter) outside of Ω so M_s is a vector multiple of the delta function, and we are interested in the accuracy of our approximation at some point (receiver) outside of Ω.

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It can be shown that the magnetic field satisfies the following integral equation

$$H(x) = H_b(x) + \gamma \nabla_x \times \int_\Omega g(x, y) q(y) E(y) dy \tag{2}$$

where

$$g(x, y) := \frac{e^{i\gamma\zeta|x-y|}}{4\pi|x - y|}$$

with $\zeta = (1 + i)/\sqrt{2}$. The function $q = \sigma - 1$. The function H_b is the background magnetic field and is assumed to be given in this situation. The well known *Born approximation* which we will also consider is formed by substituting the given background electric field E_b in the right side for E;

$$H_B(x) = H_b(x) + \gamma \nabla_x \times \int_\Omega g(x, y) q(y) E_b(y) dy. \tag{3}$$

The SLN approximation has the form

$$H_{SLN}(x) = H_b(x) + \gamma \nabla_x \times \int_\Omega g(x, y) q(y) \Gamma_0 E_b(y) dy$$

where

$$\Gamma_0 := \frac{3\sigma_b}{2\sigma_b + \sigma_s} I$$

A discussion of the motivation for this new approximation is given in section 3.

The paper [3] which introduces the SLN approximation also provides computational results in the case Ω is a sphere and M_s is a delta function. These numerics give evidence that the SLN and Born approximations are accurate if σ_s/σ_b, ω, and D are of moderate size and the parameters $\rho_R = \text{dist}(x_R, \Omega)$ and $\rho_T = \text{dist}(x_T, \Omega)$ are large ($\rho_R > 1$ and $\rho_T > 1$) where x_R and x_T are the receiver and transmitter locations, respectively. The computations also indicate that when the contrast σ_s/σ_b is large the Born approximation fails while the SLN remains reasonably accurate. Our results in this paper support these conclusions and provide some theoretical reasons for these observations. Note that throughout this paper C will represent an $O(1)$ positive constant.

We now give a brief outline and summarize our main results. In section 2 we describe the nondimensionalization and give sample values for the various parameters. In section 3 a derivation of the SLN approximation is presented. Section 4 contains some basic a priori estimates on the L^2-norm of the scattered E-field. In section 5 we show the following:

$$|(H - H_B)(x_R)| \le CQ\gamma\rho_R^{-1}\rho_T^{-1} \quad \text{and} \quad |(H - H_{SLN})(x_R)| \le CQ\gamma\rho_R^{-1}\rho_T^{-1} \tag{4}$$

where $Q = q|_\Omega = (\sigma_s - \sigma_b)/\sigma_b$. Both of these upper bounds "blow up" as $Q \to \infty$. This is an appropriate portrayal of the behavior of the Born approximation (see [3]). The SLN approximation, however, is more robust. In section 6 we give a nonrigorous asymptotic argument that $E \to 0$ as $Q \to \infty$ in the scatterer and this allows us to argue, in section 7, that

$$|(H - H_{SLN})(x_R)| \leq C\rho_R^{-1}\rho_T^{-1} \tag{5}$$

when Q is large which is our other main result.

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2. Nondimensionalization:

The time harmonic Maxwell equations which we study, before nondimensionalization, are as follows

$$\tilde{\nabla} \times \tilde{E} - i\omega\mu_b\tilde{H} = -\tilde{M}_s(\tilde{x}) \quad \text{in} \quad \mathbf{R}^3 \tag{6}$$

$$\tilde{\nabla} \times \tilde{H} - \tilde{\sigma}(\tilde{x})\tilde{E} = 0 \quad \text{in} \quad \mathbf{R}^3 \tag{7}$$

where $\tilde{E}$ is the electric field, $\tilde{H}$ is the magnetic field, $\omega > 0$ is the frequency, μ_b is the background magnetic permittivity, $\tilde{M}_s$ is the impressed magnetic source, and we have set the background electric permittivity equal to zero since it is typically negligibly small ($\epsilon_b = 8.854 \times 10^{-12}\ f/m$ where $f =$ Farads and $f/m = C^2/(N\ m^2)$). The conductivity has the special form

$$\tilde{\sigma}(\tilde{x}) = \begin{cases} \sigma_s & \text{if } \tilde{x} \in \tilde{\Omega} \\ \sigma_b & \text{if } \tilde{x} \in \mathbf{R}^3 - \tilde{\Omega}. \end{cases}$$

where the constant $\sigma_s > 0$ is the conductivity in the scatterer which is the set $\tilde{\Omega}$ that is a bounded simply connected region with diameter D and a smooth boundary. The constant $\sigma_b > 0$ is the constant background conductivity. Primarily for simplicity we will assume that $\sigma_s \geq \sigma_b$. We assume $\tilde{M}_s$ has compact support in $\mathbf{R}^3 - \tilde{\Omega}$.

Below is a table with some specific values for the constants as suggested in [3].

Name	Value	Units
Frequency	$\omega = 100\ Hz$	$Hz = s^{-1}$
Scatterer Diameter	$D = 30\ m$	$m =$ meters
Background Conductivity	$\sigma_b = 0.1\ S/m$	$S/m = C^2/(N\ m^2\ s)$
Background Permittivity	$\mu_b = 1.2 \times 10^{-6}\ h/m$	$h/m = (N\ s^2)/(C^2)$
Electric Field	$E_c = 10^{-10}\ v/m$	$v/m = N/C$
Magnetic Field	$H_c = 3 \times 10^{-9}\ a/m$	$a/m = C/(ms)$
Magnetic Source	$M_{s,c} = 3.3 \times 10^{-11}\ v/m^2$	$v/m^2 = N/(Cm)$
Transmitter Location	$\tilde{\rho}_T = 60\ m$	$m =$ meters
Receiver Location	$\tilde{\rho}_R = 60\ m$	$m =$ meters

Variables and Units ($C =$ Coulombs, $s =$ seconds, $N =$ Newtons, $Hz =$ Hertz, $S =$ Siemens, $h =$ henrys, v = volts, and $a =$ amps).

Note that $\tilde{\rho}_R = \text{dist}(\tilde{x}_R, \tilde{\Omega})$ and $\tilde{\rho}_T = \text{dist}(\tilde{x}_T, \tilde{\Omega})$ where $\tilde{x}_R$ and $\tilde{x}_T$ are the receiver and transmitter locations, respectively. We now make the following variable changes to simplify,

$$\gamma = \sqrt{\omega\mu_b\sigma_b}D, \quad \tilde{x} = Dx, \quad \tilde{E}(\tilde{x}) = E_c E(x),$$

$$\tilde{M_s}(\tilde{x}) = M_{s,c}\, M_s(x), \quad \tilde{\sigma}(\tilde{x}) = \sigma_b\, \sigma(x), \quad \text{and} \quad \tilde{H}(\tilde{x}) = H_c\, H(x)$$

where we relate

$$H_c = \sigma_b D\gamma^{-1}E_c \quad \text{and} \quad E_c = DM_{s,c}.$$

We also define

$$\rho_T = \tilde{\rho}_T/D \quad \text{and} \quad \rho_R = \tilde{\rho}_R/D.$$

Using these substitutions in (6)–(7) we obtain the equations (1). Note that $\gamma = 0.104$ so that our assumption on the size of γ is satisfied.

Other forms of the Maxwell equations will be useful in our analysis. Taking the curl of one of the equations and substituting in the other we obtain

$$\nabla \times \nabla \times E - i\gamma^2\sigma E = -\nabla \times M_s. \tag{8}$$

$$\nabla \times \left(\sigma^{-1}\nabla \times H\right) - i\gamma^2 H = \gamma M_s. \tag{9}$$

We now split the fields into the background and scattered fields $E = E_b + E_s$ and $H = H_b + H_s$ where from (1) we have

$$\nabla \times E_b - i\gamma H_b = -M_s \quad \text{and} \quad \nabla \times H_b - \gamma E_b = 0 \quad \text{in} \quad \mathbf{R}^3 \tag{10}$$

and

$$\nabla \times E_s - i\gamma H_s = 0 \quad \text{and} \quad \nabla \times H_s - \gamma\sigma E_s = \gamma q E_b \quad \text{in} \quad \mathbf{R}^3. \tag{11}$$

Again taking the curl of one equation and substituting in the other we have

$$\nabla \times \nabla \times E_s - i\gamma^2\sigma E_s = i\gamma^2 q E_b. \tag{12}$$

$$\nabla \times \left(\sigma^{-1}\nabla \times H_s\right) - i\gamma^2 H_s = \gamma\nabla \times \left(\frac{q}{\sigma}E_b\right). \tag{13}$$

For later use let $Q = q\,|_\Omega = (\sigma_s - \sigma_b)/\sigma_b$ be the conductivity contrast.

3. The Born and Static Localized Nonlinear Approximations:

In this section we describe the well-known Born approximation and introduce the SLN approximation of [3]. To accomplish this we must describe the Green's function representation of the electric field solution of (12). The function g satisfies

$$\Delta_y g(x, y) + i\gamma^2 g(x, y) = -\delta(x - y)$$

and from this (see Kong [5]) we find that

$$G(x,y) = (I + (i\gamma^2)^{-1}\nabla_x^2)g(x,y) \quad \text{satisfies} \quad \nabla_y \times \nabla_y \times G - i\gamma^2 G = -\delta(x-y)I.$$

It can be shown using the properties of g and G that

$$E_s(x) = (i\gamma^2 I + \nabla_x^2)\int_\Omega g(x,y)q(y)E(y)dy.$$

Then

$$E(x) = E_b(x) + (i\gamma^2 I + \nabla_x^2)\int_\Omega g(x,y)q(y)E(y)dy. \tag{14}$$

One can now derive (2) from the above and the Maxwell equations (1).

To introduce the new approximation from Habashy, Groom, and Spies we find, by rewriting (14), that

$$E(x) = E_b(x) + \left[(i\gamma^2 I + \nabla_x^2)\mathcal{G}(x)\right]E(x) + T(x) \tag{15}$$

where

$$T(x) = (i\gamma^2 I + \nabla_x^2)\int_\Omega g(x,y)q(y)(E(y) - E(\bar{x}))dy\Big|_{\bar{x}=x},$$

and

$$\mathcal{G}(x) = \int_\Omega g(x,y)q(y)dy.$$

We then have

$$E = \Gamma(E_b + T).$$

where

$$\Gamma^{-1}(x) = I - (i\gamma^2 I + \nabla_x^2)\mathcal{G}(x).$$

Substituting this expression for E in (2) we obtain

$$H = H_{LN} + e$$

where H_{LN} is the *localized nonlinear* approximation;

$$H_{LN}(x) = H_b(x) + \gamma\nabla_x \times \int_\Omega g(x,y)q(y)\Gamma(y)E_b(y)dy;$$

and the error term is

$$e(x) = \gamma\nabla_x \times \int_\Omega g(x,y)q(y)\Gamma(y)T(y)dy.$$

The terminology is motivated by the *localization* that occurs in (15) and the fact that H_{LN} is not linear in q (since Γ depends on q) while the typical algorithms such as Born are linear in q.

Finally, we define the SLN approximation. This is introduced in [3] by examining the Γ function defined above in the case where γ is small. They find that $\Gamma \cong \Gamma_0$ and then define H_{SLN} from the above.

4. A Priori Estimate for E:

In this section we derive an L^2-estimate for the scattered electric field where the bound depends on the L^2-norm of the background field E_b. This estimate will be useful in our error analysis.

Since the background conductivity is nonzero in all of $\mathbf{R}^3$ it follows that the solutions to our problem will decay exponentially as $|x| \to \infty$. This is clear from examining the Green's function formulations (see (14) and (2)). It is natural to require

$$[E \times \nu] = 0 \quad \text{and} \quad [H \times \nu] = 0 \quad \text{on } \partial\Omega \tag{16}$$

where

$$[v] = v\,|_{\text{Limit from inside } \Omega} - v\,|_{\text{Limit from outside } \Omega} \,.$$

and ν is the outward pointing unit normal from Ω to $\partial\Omega$. These conditions imply that the traces of the tangential components of E and H are continuous. We assume that there exists a unique solution consisting of E and H fields which satisfy the Maxwell equations, decay exponentially for large $|x|$ and satisfy the conditions (16) (see [1] for a proof of existence).

Complementary to the conditions (16) are the following conditions shown in [1] on the normal components:

$$[\sigma E \cdot \nu] = 0 \text{ and } [H \cdot \nu] = 0 \text{ on } \partial\Omega. \tag{17}$$

Note that the statements (16) and (17) apply to E_s and H_s since the background fields are smooth across $\partial\Omega_\gamma$. Also observe that (16) and (17) imply that H and H_s are continuous across $\partial\Omega$.

We will need to evaluate the divergence of the E and H fields. If we compute this quantity in Ω and $\mathbf{R}^3 - \Omega$ separately and note that $\nabla q(x) = 0$ in these regions we have from (1)

$$\nabla \cdot E_s(x) = 0 \quad \text{and} \quad \nabla \cdot H_s(x) = 0 \tag{18}$$

(Note that $\nabla \cdot E_b = 0$).

The Green identity

$$\int_V (\nabla \times F) \cdot L \, dx = \int_V F \cdot (\nabla \times L) \, dx + \int_{\partial V} (F \times L) \cdot \nu \, ds, \tag{19}$$

will allow us to derive a variational formula from which we can obtain the desired estimate. We will use the arithmetic-geometric mean inequality,

$$ab \leq \beta a^2 + \frac{1}{4\beta} b^2, \quad (0 < \beta < 1). \tag{20}$$

We are now in a position to derive the estimate for E_s. We first take the dot product of the defining equation for E_s, (12), with the conjugate $\bar{E}_s$, and apply (19) in Ω and $\mathbf{R}^3 - \Omega$ separately. This yields

$$\left(\int_\Omega + \int_{\mathbf{R}^3\setminus\Omega}\right) |\nabla \times E_s|^2 \, dx + \int_{\partial\Omega} \left[\nabla \times E_s \times \bar{E}_s \cdot \nu\right] \, ds - i\gamma^2 \int_{\mathbf{R}^3} \sigma |E_s|^2 \, dx = i\gamma^2 \int_\Omega q E_b \cdot \bar{E}_s \, dx.$$

Since $\nabla \times E$ is not defined on $\partial\Omega$ we must integrate it over Ω and $\mathbf{R}^3 - \Omega$ separately. For this we use the more concise notation

$$\left(\int_\Omega + \int_{\mathbf{R}^3\setminus\Omega}\right) f dx = \int_\Omega f dx + \int_{\mathbf{R}^3 - \Omega} f dx.$$

Using (11) to rewrite the boundary term and the identity $H_s \times \bar{E}_s \cdot \nu = \bar{E}_s \times \nu \cdot H_s$. We have

$$\left(\int_\Omega + \int_{\mathbf{R}^3\setminus\Omega}\right) |\nabla \times E_s|^2 \, dx - i\gamma^2 \int_{\mathbf{R}^3} \sigma |E_s|^2 \, dx + i\gamma \int_{\partial\Omega} \left[\bar{E}_s \times \nu \cdot H_s\right] \, ds = i\gamma^2 \int_\Omega q E_b \cdot \bar{E}_s \, dx.$$

The boundary term vanishes due to (16) and the fact that H_s is continuous across $\partial\Omega$. Now, taking real and imaginary parts and using (20) we obtain

$$\left(\int_\Omega + \int_{\mathbf{R}^3\setminus\Omega}\right) |\nabla \times E_s|^2 \, dx + \gamma^2 \int_{\mathbf{R}^3} \sigma |E_s|^2 \, dx = \gamma^2 \int_\Omega q |E_b|^2 \, dx. \tag{21}$$

5. Estimates for the Born and SLN Approximations:

In this section we furnish the estimates (4) for the accuracy of the Born and SLN approximations in the case when the constrast σ_s/σ_b is of moderate size. In this case we note that from (21) we have

$$\|E_s\|_{L^2(\Omega)} \leq \|E_b\|_{L^2(\Omega)} \tag{22}$$

We assume that E_b on Ω satisfies the estimate

$$\|E_b\|_{L^\infty(\Omega)} \leq C\rho_T^{-1}. \tag{23}$$

This is reasonable if, as in [3], we have $M_s = \delta(\cdot - x_T)U_b$ where U_b is a vector with $|U_b| = O(1)$ and then $E_b = -\nabla g(\cdot, x_T) \times U_b$. It then follows that

$$E_b(x) = \frac{-e^{i\gamma\zeta|x-x_T|}}{4\pi} \left(\frac{i\gamma\zeta}{|x-x_T|} - \frac{1}{|x-x_T|^2}\right) \frac{x-x_T}{|x-x_T|} \times U_b$$

and thus for $x \in \Omega$

$$|E_b(x)| \leq \frac{1}{4\pi} \left(\frac{\gamma}{\rho_T} + \frac{1}{\rho_T^2}\right) |U_b|$$

from which (23) follows, as a reasonable hypothesis, since $\gamma < 1$ and $\rho_T \geq 1$. Observe that a similar argument gives

$$|\nabla g(x_R, y)| \leq C\rho_R^{-1}. \tag{24}$$

for $y \in \Omega$. We are now in a position to estimate the accuracy of the Born approximation. Subtracting (3) from (2) and evaluating the resulting equation at $x = x_R$ we have

$$|(H - H_B)(x_R)| \leq \gamma Q \left| \int_\Omega \nabla g(x_R, y) \times (E - E_b)(y) dy \right|.$$

From the estimate (24) on $|\nabla g|$ we have

$$|(H - H_B)(x_R)| \leq CQ\gamma\rho_R^{-1} \|E_s\|_{L^2(\Omega)}$$

and applying (22) we obtain

$$|(H - H_B)(x_R)| \leq CQ\gamma\rho_R^{-1} \|E_b\|_{L^2(\Omega)} \leq CQ\gamma\rho_R^{-1}\rho_T^{-1}.$$

This gives the first part of (4). Note that the estimate "blows up" in the case when Q is large which corresponds to the contrast σ_s/σ_b also being large.

The estimation of the accuracy of the SLN approximation is quite similar and leads to the same result which is the second part of (4). Here we have

$$|(H - H_{SLN})(x_R)| \leq \gamma Q \left| \int_\Omega \nabla g(x_R, y) \times (E - \Gamma_0 E_b)(y) dy \right|$$

and the integrand can be decomposed as

$$E - \Gamma_0 E_b = E_s + (I - \Gamma_0) E_b.$$

Thus the estimate will involve two terms, the one with the scattered field which is essentially the Born approximation error term and the second term will have the background field and the $I - \Gamma_0$ factor. So

$$\begin{aligned} |(H - H_{SLN})(x_R)| &\leq CQ\gamma\rho_R^{-1} \left(\|E\|_{L^2(\Omega)} + \left| 1 - \frac{3\sigma_b}{2\sigma_b + \sigma_s} \right| \|E_b\|_{L^2(\Omega)} \right) \\ &\leq CQ\gamma\rho_R^{-1} \left(1 + \left| \frac{\sigma_s - \sigma_b}{2\sigma_b + \sigma_s} \right| \right) \|E\|_{L^2(\Omega)} \end{aligned}$$

From which the second part of (4) follows.

6. Perturbation Approximation of E for High Contrasts:

The numerical experiments in [3] (see figures 3 and 4 on pages 1765 and 1766) indicate that the SLN approximation is accurate in the high conductivity case when the Born approximation generally fails. The key reason is that the tensors Γ and Γ_0 tend to zero as $Q \to \infty$ (this is noted in section 4.1 of [7]). In this section we focus on the behavior of the electric field near the boundary when $\epsilon := \sqrt{\sigma_b/\sigma_s}$ is small (equivalent to Q large since $\epsilon = (Q+1)^{-1/2}$) and Ω is a sphere. We give a non-rigorous asymptotic argument that $E = 0$ over most of the region occupied by the scatterer with a thin layer near the boundary. We will also see that $E = O(\epsilon)$ on that boundary and decays exponentially in the layer.

Let S denote the electric field inside the scatterer and B outside. From (8) and the identity

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \Delta A. \tag{25}$$

we have that

$$\Delta S + i\epsilon^{-2}\gamma^2 S = 0 \text{ and } \nabla \cdot S = 0 \quad \text{in } \Omega, \tag{26}$$

$$\Delta B + i\gamma^2 B = -\nabla \times M_s \text{ and } \nabla \cdot B = 0 \quad \text{in } \mathbf{R}^3 \setminus \Omega, \tag{27}$$

and conditions

$$(S - B) \times \nu = 0 \text{ and } (\nabla \times (S - B)) \times \nu = 0 \text{ on } \partial\Omega \tag{28}$$

$$B(x) \to 0 \text{ as } |x| \to 0. \tag{29}$$

We first determine the outer solution in the scatterer and thus look for S in the form

$$S = S^0 + \epsilon S^1 + \dots .$$

Substituting in (26) we obtain, after some rearrangement,

$$i\epsilon^{-2}\gamma^2 S^0 + i\epsilon^{-1}\gamma^2 S^1 + (\Delta S^0 + i\gamma^2 S^2) + \epsilon(\Delta S^1 + i\gamma^2 S^3) + \dots = 0.$$

Setting the coefficients of the differing powers of ϵ on the left side to zero we find that $S^0 = S^1 = S^2 = \dots = 0$. So to match the B and S functions we develop a boundary layer solution.

We now change to spherical coordinates (r, θ, ϕ). Define η by the equation $r = 1 - \epsilon\eta$ and $W(\eta, \theta, \phi) = S(r, \theta, \phi)$. The function W will provide the behavior of the electric field, S, near $\partial\Omega$ in Ω. Since

$$\frac{\partial S}{\partial r} = -\frac{1}{\epsilon}\frac{\partial W}{\partial \eta}$$

the Helmholz equation in (26) becomes

$$\frac{1}{\epsilon^2}\left(\frac{\partial^2 W}{\partial \eta^2} + i\gamma^2 W\right) - \frac{2}{\epsilon}\left[\frac{\partial}{\partial \eta}\left(\eta \frac{\partial W}{\partial \eta}\right) + i\eta\gamma^2 W\right] + O(1) = 0. \tag{30}$$

From the divergence condition in (26) we have

$$-\frac{1}{\epsilon}\frac{\partial W_r}{\partial \eta} + \left[2\frac{\partial}{\partial \eta}(\eta W_r) + \nabla_T \cdot W_r\right] + O(\epsilon) = 0 \tag{31}$$

where

$$W = \begin{pmatrix} W_r \\ W_\theta \\ W_\phi \end{pmatrix}, \quad W_T = \begin{pmatrix} W_\theta \\ W_\phi \end{pmatrix},$$

and

$$\nabla_T \cdot A_T = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{\partial A_\phi}{\partial\theta}\right).$$

The first part of the boundary condition (28) implies that $W_T = B_T$ while the second part gives

$$\frac{1}{\epsilon}\frac{\partial}{\partial\eta}\begin{pmatrix} 0 \\ W_\theta \\ W_\phi \end{pmatrix} + \begin{pmatrix} 0 \\ W_\theta - \partial W_r/\partial\theta \\ W_\phi - (\sin\theta)^{-1}\partial W_r/\partial\phi \end{pmatrix} + O(\epsilon) = \begin{pmatrix} 0 \\ B_\theta - \partial B_r/\partial\theta \\ B_\phi - (\sin\theta)^{-1}\partial B_r/\partial\phi \end{pmatrix} \tag{32}$$

at the boundary. Choosing

$$W = W^0 + \epsilon W^1 + \dots$$

we find from (30)–(32) that

$$\frac{1}{\epsilon^2}\left(\frac{\partial^2 W^0}{\partial\eta^2} + i\gamma^2 W^0\right) + \frac{1}{\epsilon}\left[\frac{\partial^2 W^1}{\partial\eta^2} + i\gamma^2 W^1 - 2\left(\frac{\partial}{\partial\eta}\left(\eta\frac{\partial W^0}{\partial\eta}\right) + i\eta\gamma^2 W^0\right)\right] + O(1) = 0, \tag{33}$$

$$-\frac{1}{\epsilon}\frac{\partial W_r^0}{\partial\eta} + \left[-\frac{\partial W_r^1}{\partial\eta} + 2\frac{\partial}{\partial\eta}\left(\eta W_r^0\right) + \nabla_T\cdot W_r^0\right] + O(\epsilon) = 0, \tag{34}$$

and at the boundary

$$\frac{\partial}{\partial\eta}\left(\frac{1}{\epsilon}W_T^0 + W_T^1\right) + W_T^0 - \begin{pmatrix} \partial W_r^0/\partial\theta \\ (\sin\theta)^{-1}\partial W_r^0/\partial\phi \end{pmatrix} + O(\epsilon) = B_T - \begin{pmatrix} \partial B_r/\partial\theta \\ (\sin\theta)^{-1}\partial B_r/\partial\phi \end{pmatrix}. \tag{35}$$

We are now in a position to give a description of the zeroth order solution. From (34) we have that

$$\frac{\partial W_r^0}{\partial\eta} \equiv 0.$$

Since this implies $\partial^2 W_r^0/\partial\eta^2 \equiv 0$ we have from (33) that $W_r^0 \equiv 0$. Also from (33) we can solve for the leading order tangential terms and obtain

$$W_T^0(\eta,\theta,\phi) = A_T^0(\theta,\phi)e^{-\zeta\gamma\eta}$$

where we dropped the growing exponential term to match the outer solution. Now from (35) we can conclude $-\zeta A_T^0 \equiv 0$ so

$$W^0 \equiv 0.$$

We now can provide a boundary condition for the B function from the first part of (28) since $S \times \nu = O(\epsilon)$ and thus

$$B \times \nu \cong 0. \tag{36}$$

We now determine the first order solution. Substituting the zeroth order solution, W^0 in (33) we find

$$W^1(\eta, \theta, \phi) = A^1(\theta, \phi) e^{-\zeta\gamma\eta}$$

from (34) we find that

$$A_r^1 \equiv 0.$$

Now from (35) we have

$$B_T - \begin{pmatrix} \partial B_r/\partial\theta \\ (\sin\theta)^{-1}\partial B_r/\partial\phi \end{pmatrix} = \frac{\partial W_T^1}{\partial\eta} = -\gamma\zeta A_T^1(\theta, \phi)$$

on the boundary. Thus

$$|W_T^1| = \frac{1}{\gamma}(|B_T| + |\nabla_T B_r|) e^{-\gamma\eta/\sqrt{2}}$$

where B_T and $\nabla_T B_r$ are evaluated on the boundary. We, therefore, need to estimate the size of B_T and $\nabla_T B_r$. To obtain a reasonable approximation we again consider the model case where $M_s = \delta(\cdot - x_T)U_b$. Since the function $-\nabla g(\cdot, x_T) \times U_b$ satisfies (27) and (29) and has magnitude $\gamma\rho_T^{-1} \cong 0$ on $\partial\Omega$ we take

$$B \cong \nabla g(\cdot, x_T) \times U_b.$$

A short calculation shows that if $\rho_T >> 1$ then

$$|B_T| + |\nabla_T B_r| \leq \gamma\rho_T^{-1}.$$

We can now put all these results together to describe E in the case of high conductivity contrast σ_s/σ_b. We have found, from the outer solution, that $E = 0$ over most of the domain Ω except for an $O(\epsilon)$ boundary layer where

$$E \cong \epsilon\rho_T^{-1} e^{-\gamma(1-r)/(\sqrt{2}\epsilon)}$$

and thus the $L^1(\Omega)$-norm of E is

$$\int_\Omega |E| dx = \frac{C\epsilon^2}{\gamma\rho_T} \tag{37}$$

where C is an $O(1)$ constant.

7. Born and SLN Approximations at High Contrasts:

As we have seen in section 6 the electric field tends to zero in the scatterer as $Q \to \infty$. In the Born approximation the key replacement is the substitution of E_b for E. Since $E_b \neq 0$ we expect the Born approximation will blow-up when the contrast becomes large.

The key replacement in the SLN approximation is the substitution of $\Gamma_0 E_b$ for E. Since

$$\Gamma_0 E_b = \frac{3}{3+Q} E_b \to 0 \text{ as } Q \to 0$$

we expect the SLN approximation will be accurate since the $\Gamma_0 E_b$ term models the behavior of E in the case of large Q. In rest of this section we use the result of section 6 to make this statement more precise.

From (37) and the definition of ϵ in that section we have

$$\|E\|_{L^1(\Omega)} \leq \frac{C}{\gamma \rho_T Q}.$$

Then since $|\nabla g| \leq C\rho_R^{-1}$, $|E_b| \leq C\rho_T^{-1}$, and the coefficient of I in Γ_0 is $O(Q^{-1})$, we have

$$\begin{aligned} |(H - H_{SLN})(x_R)| &\leq \gamma Q \int_\Omega |\nabla g(x_R, y)||E(y) - \Gamma_0 E_b(y)|dy \\ &\leq C\gamma \rho_R^{-1} Q \left(\frac{1}{\gamma \rho_T Q} + \frac{1}{Q\rho_T} \right) \\ &\leq C\rho_R^{-1} \rho_T^{-1}. \end{aligned}$$

This estimate is (5). Thus the SLN approximation will remain accurate in the high contrast case.

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