Addendum to [2]

W. Bryc

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Abstract

This note examines accuracy of some other approximations, overlooked in [2].

Firstly, we corrects an erroneous formula attributed in [2, page 372] to Hastings [3]. Hastings [3] gives several other approximations that are more accurate than the one cited. The tables below list illustrative values.

Secondly, we present Boyd's [1] bounds for the Mills ratio, and again we examine those numerically.

Thirdly, we present Dorenzo's [5] integer approximation, and again analyze it numerically.

1 Hastings [3]

This section deals with various approximations given by Hastings. We used Mathematica to convert Hastings formulas into the form conforming to [2]. This resulted in displays of only 6-digits accuracy even if the original constants had more digits. For actual computations, Hastings constants were used with his precision, which is typically higher than what is displayed in this note.

1.1 [3, page 185, Sheet 61]

This approximation was misquoted in [2]; this approximation is cited also in [4, page 51, Section 3.2.2]. Translated to normal tails, it reads

$$\Pr(Z > z) \approx \frac{1}{2\left(1 + 0.196854\, z + 0.115195\, z^2 + 0.000343654\, z^3 + 0.019527\, z^4\right)^4} \tag{1}$$

Here is the same formula within Mathematica arithmetic − this is the form we use below in actual computations.

$$\frac{1}{2\left(0.019527 z^4 + 0.00034365389565666214 z^3 + 0.1151945 z^2 + 0.19685357813486654 z + 1\right)^4}$$

This approximation is not uniform; relative error increases with z. It should not be used for $z > 3$.

1.2 [3, page 167, Sheet 43]

$$\Pr(Z > z) \approx \frac{13.5806 + 2.7115\,z + 0.523073\,z^2}{e^{\frac{z^2}{2}}\,(3.00596 + 1.\,z)^3} \tag{2}$$

This approximation is not uniform; relative error increases linearly with z. But it performs respectably for large z. For example, $\Pr(Z > 11.8) = 1.952 \times 10^{-32} \approx 2.12 \times 10^{-32}$.

1.3 [3, page 168, Sheet 44]

$$\Pr(Z > z) \approx \frac{0.266784\,(23.1544 + z)\,\left(15.2105 + 3.63895\,z + z^2\right)}{e^{\frac{z^2}{2}}\,(3.70247 + 1.\,z)^4} \tag{3}$$

This approximation is not uniform; relative error increases linearly with z. But it performs respectably for large z. For example, $\Pr(Z > 11.8) = 1.952 \times 10^{-32} \approx 1.852 \times 10^{-32}$.

[3, page 169, Sheet 45]

$$\Pr(Z > z) \approx \frac{0.550051\,\left(12.4037 + 3.45399\,z + z^2\right)\,\left(109.883 + 8.99445\,z + z^2\right)}{e^{\frac{z^2}{2}}\,(4.31701 + 1.\,z)^5} \tag{4}$$

This approximation is not uniform; relative error increases linearly with z. But it performs respectably for large z. For example, $\Pr(Z > 11.8) = 1.952 \times 10^{-32} \approx 2 \times 10^{-32}$.

2 Boyd [1]

Boyd [1] gives two bounds which are good uniform approximations to the tail integrals.

2.1 Lower bound

$$\Pr(Z > z) \geq \frac{\sqrt{\pi/2}}{\sqrt{z^2 + 2\pi} + (\pi - 1)z} e^{-z^2/2} \tag{5}$$

For example, $\Pr(Z > 11.8) = 1.952 \times 10^{-32} \geq 1.9515 \times 10^{-32}$.

2.2 Upper bound

$$\Pr(Z > z) \leq \frac{\sqrt{\pi/2}}{\sqrt{(\pi - 2)^2 z^2 + 2\pi} + 2z} e^{-z^2/2} \tag{6}$$

For example, $\Pr(Z > 11.8) = 1.952 \times 10^{-32} \leq 1.9532 \times 10^{-32}$.

3 Derenzo [5]

According to information from Tony Hutchins, Derenzo's approximation is

$$\Pr(Z > z) \approx .5\exp(-((83x + 351)x + 562)/(703/x + 165)) \qquad (7)$$

This approximation is not uniform, but performs respectably for large z. For example, $\Pr(Z > 11.8) = 1.952 \times 10^{-32} \approx 1.792010^{-32}$.

4 Numerical comparison

The following table complements [2, Tables 1 and 2] (here: six-digits only).

Table 1: Illustration of the accuracy of Hastings, Boyd and Derenzo approximate formulae for the normal tail probabilities $\Pr(Z > z)$ (small z).

Approx.	z=0.1	z=0.5	z=1.0	z=1.5
Exact [2]	0.4601721627	0.3085375387	0.1586552539	0.0668072013
(1) [3]	0.460405	0.308305	0.158876	0.0667299
(2) [3]	0.460161	0.308548	0.158648	0.0668016
(3) [3]	0.4601713	0.3085381	0.1586545	0.0668079
(4) [3]	0.46017211	0.308537537	0.158655263	0.06680723
(5) [1]	0.458	0.305	0.1571	0.0663
(6) [1]	0.4603	0.3097	0.1599	0.0674
Ave of (5), (6)	0.4592	0.3074	0.1585	0.06688
(7) [5]	0.460127927598	0.308571547840	0.1587201614338	0.0668275000959

Table 2: Illustration of the accuracy of Hastings, Boyd and Derenzo approximate formulae for the normal tail probabilities $\Pr(Z > z)$ (large z).

Approx.	z=2.0 ($\times 10^{-2}$)	z=3.0 ($\times 10^{-3}$)	z=5.0 ($\times 10^{-7}$)	z=7.0 ($\times 10^{-12}$)	z=9.0 ($\times 10^{-19}$)
Exact [2]	2.275013195	1.349898032	2.866515719	1.279812544	1.128588406
(1) [3]	2.25631	1.57917	58.3205	54543.	(+)
(2) [3]	2.27587	1.3549	2.92056	1.33006	1.19642
(3) [3]	2.27503	1.34923	2.8504	1.25913	1.09491
(4) [3]	2.275006	1.34997	2.87105	1.28813	1.14519
(5) [1]	2.26	1.347	2.865	1.2796	1.12852
(6) [1]	2.295	1.359	2.878	1.28	1.13
Ave of (5), (6)	2.2798	1.353	2.871	1.28	1.129
(7) [5]	2.27503	1.34952	2.86687	1.2731	1.101

Table 3: Illustration of the accuracy of Hastings and Derenzo approximate formulae for the normal tail probabilities $\Pr(Z > z)$ (tiny z).

z	0.001	0.01	0.1	0.2	0.3	0.4	0.5	0.6
Mathematica	0.499601	0.496011	0.460172	0.42074	0.382089	0.344578	0.308538	0.274253
(4) [3]	0.499601	0.496011	0.460172	0.42074	0.382089	0.344578	0.308538	0.274253
(7) [5]	0.4996	0.496003	0.460128	0.420697	0.382067	0.344586	0.308572	0.274307

4.1 Small z comparison

5 Conclusions

The series expansion on a 32-bit computer is most accurate in the range of $0 < z < 3$. Strecock-Moran approximation cited as [2, (9)] is extremely good for $0 < z < 5$. Hastings best approximation (4) is a bit less accurate, but simpler computationally. Boyd bounds are low complexity approximations that work well for large z. Derenzo's formula is very simple (integer coefficients only) yet performs suprisingly well accross a wide range of z,

Our comparison indicates that for programming applications one should switch between two approximations: Hart's uniform approximation cited as [2, (11)] outperforms other approximations for $z > 3$, and series expansion outperforms other methods for small $0 < z < 3$.

Derenzo's approximation (7) is a top approximation for "calculator" applications, with unbeatable combination of simplicity and performance accross the range of $0 < z < 12$.

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References

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