A REMARK ON THE MAXIMUM EIGENVALUE FOR CIRCULANT MATRICES

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Abstract. We point out that the method of Davis-Mikosch [1] gives for a symmetric circulant $n \times n$ matrix composed of i.i.d. entries with mean 0 and finite $(2+\delta)$-moments in the first half-row that the maximum eigenvalue is on the order $\sqrt{2n\log n}$, and the fluctuations are Gumbel.

Let $\{X_0, X_1, \ldots\}$ be i.i.d. mean-zero, variance 1, random variables. For $m \geq 1$, consider the $(2m+1) \times (2m+1)$ "palindromic" circulant matrix

$$\begin{bmatrix} X_0 & X_1 & X_2 & \cdots & X_m & X_m & X_{m-1} & \cdots & X_1 \\ \vdots & & & & \vdots & & & & \vdots \\ X_m & X_{m-1} & \ldots & & X_0 & X_1 & X_2 & \ldots & X_m \\ \vdots & & & & \vdots & & & & \vdots \\ X_1 & X_2 & X_3 & \cdots & X_m & X_{m-1} & X_{m-2} & \cdots & X_0 \end{bmatrix}. \tag{1}$$

In this note, we observe, for circulant matrices (1), that an argument of [1] for the maximum of periodograms easily applies to deduce that the maximum eigenvalue is on the order $\sqrt{2m\log m}$, and the fluctuations are Gumbel (Theorem 1). In particular, a sort of "universality" with respect to the entries $\{X_i\}$, much discussed in other contexts in the random matrix literature, is established for the asymptotic maximum eigenvalue distribution. We refer to [3] for more discussion of random circulant matrices, and note the result for Gaussian entries is as well given in [3, Corollary 5].

Theorem 1. *Suppose $X_1, X_2, \ldots$ are i.i.d. with $E(X_1) = 0$, $E(X_1^2) = 1$, and $E(|X_1|^s) < \infty$ for some $s > 2$. Denote by λ_m the maximum eigenvalue of (1), and let $a_m = \sqrt{2\log m} - \log(4\pi \log m)/\big(2\sqrt{2\log m}\big)$. Then*

$$\lim_{m\to\infty} P\left(\left(\frac{\lambda_m}{\sqrt{2m+1}} - a_m\right)\sqrt{2\log m} \leq x\right) \;=\; G(x)$$

where $G(x) = \exp(-e^{-x})$.

The proof follows closely the method used to prove [1, Theorem 2.1] which is based on Einmahl's multivariate extension of the Komlos-Major-Tusnady theorem (cf. Lemma 3). Indeed, Lemmas 4, 5 are similar to [1, Lemmas 3.3, 3.4] with analogous proofs. The well known Bonferroni inequalities (Lemma 2) and Lemma 3 are stated as [1, Lemmas 3.1, 3.2].

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Lemma 2. *Let $A_1,\dots,A_n$ be measurable events. Then for every $1\le k\le \lfloor n/2\rfloor$,*

$$\sum_{d=1}^{2k}(-1)^{d-1}S_d \le P(A_1\cup\cdots\cup A_n)\le \sum_{d=1}^{2k-1}(-1)^{d-1}S_d,$$

where $S_d=\sum_{1\le j_1<\cdots j_d\le n}P(A_{j_1}\cap\cdots\cap A_{j_d})$.

The next statement is Einmahl's Corollary 1(b), page 31, in combination with the Remark on page 32 [2].

Lemma 3. *Let $\xi_1,\dots\xi_n$ be independent random vectors in $\mathbb{R}^d$. Assume that the moment generating function of $\{\xi_i\}$ exists in a neighborhood of the origin, and that*

$$\mathrm{cov}(\xi_1+\cdots+\xi_n)=B_nI_d,$$

where $B_n>0$ and I_d is the d dimensional identity matrix. Let η_k be independent $N(0,\sigma^2\mathrm{cov}(\xi_k))$ random vectors for $1\le k\le n$ independent of $\{\xi_i\}$, and $0<\sigma^2\le 1$. Let $\xi_k^*=\xi_k+\eta_k$ for $1\le k\le n$, and write p_n^* as the density of $B_n^{-1/2}\sum_{k=1}^n\xi_k^*$. Choose $0<\alpha<1/2$ such that

$$\alpha\sum_{k=1}^n E|\xi_k|^3\exp(\alpha|\xi_k|)\ \le\ B_n. \tag{2}$$

Let

$$\beta_n\ =\ \beta_n(\alpha)\ =\ B_n^{-3/2}\sum_{k=1}^n E|\xi_k|^3\exp(\alpha|\xi_k|). \tag{3}$$

If

$$|x|\ \le\ c_1\alpha B_n^{1/2},\quad \sigma^2\ \ge\ -c_2\beta_n^2\log\beta_n\quad\text{and}\quad B_n\ge c_3\alpha^{-2}, \tag{4}$$

where c_1,c_2,c_3 are constants depending only on d, then

$$p_n^*(x)\ =\ \phi_{(1+\sigma^2)I_d}(x)\exp(\bar T_n(x))\quad\text{with}\quad |\bar T_n(x)|\ \le\ c_4\beta_n(|x|^3+1), \tag{5}$$

where ϕ_C is the density of the d-dimensional centered Gaussian vector with covariance matrix C and c_4 is a constant depending only on d.

Let now $\{X_j\}_{j\ge 0}$ be as in Theorem 1. For $j,m\ge 0$, define $\bar X_j=\bar X_j^{(m)}=X_j1_{|X_j|\le m^{1/s}}-E(X_11_{|X_1|\le m^{1/s}})$.

Lemma 4. *We have a.s. that*

$$\frac{2}{\sqrt{2m+1}}\max_{1\le j\le m}\sum_{k=1}^m\cos\big(\frac{2\pi jk}{2m+1}\big)X_k - \frac{2}{\sqrt{2m+1}}\max_{1\le j\le m}\sum_{k=1}^m\cos\big(\frac{2\pi jk}{2m+1}\big)\bar X_k^{(m)}\ =\ O(m^{-1/2}).$$

Proof. First, we can add and subtract $(2m+1)^{-1/2}\bar X_0$ on the left-side. Since

$$1+2\sum_{k=1}^m\cos\big(\frac{2\pi jk}{2m+1}\big)\ =\ 0,$$

we can replace

$$\bar X_0+2\sum_{k=1}^m\cos\big(\frac{2\pi jk}{2m+1}\big)\bar X_k$$

with

$$X_0 1_{|X_0|\le m^{1/s}} + 2\sum_{k=1}^m \cos\big(\frac{2\pi jk}{2m+1}\big) X_k 1_{|X_k|\le m^{1/s}}.$$

Now, by Borel-Cantelli, as $\sum_t P(|X_t| > t^{1/s}) < \infty$, we have $|X_t| \le t^{1/s}$ for all $t \ge N(\omega)$ a.s. Then,

$$\begin{aligned}
\sum_{t=1}^m \big|X_t - X_t 1_{|X_t|\le m^{1/s}}\big| &= \sum_{t=1}^m |X_t| 1_{|X_t|>m^{1/s}} \\
&\le \sum_{t=1}^{N(\omega)} X_t 1_{|X_t|>m^{1/s}} + \sum_{t=N(\omega)+1}^{m} X_t 1_{|X_t|>t^{1/s}} \\
&\le \sum_{t=1}^{N(\omega)} |X_t| 1_{|X_t|>m^{1/s}} + \sum_{t>N(\omega)} X_t 1_{|X_t|>t^{1/s}} \;=\; 0
\end{aligned}$$

for $m \ge \max\{N(\omega), |X_1|^s, \ldots, |X_{N(\omega)}|^s\}$. Hence, the sums

$$\sum_{k=1}^m \cos\big(\frac{2\pi jk}{2m+1}\big) X_k \quad \text{and} \quad \sum_{k=1}^m \cos\big(\frac{2\pi jk}{2m+1}\big) X_k 1_{|X_k|\le m^{1/s}}$$

agree for all large m a.s.

We finish by noting the extra term

$$\frac{1}{\sqrt{2m+1}}[\bar{X}_0 - X_0 1_{|X_0|\,\le\, m^{1/s}}] \;=\; \frac{1}{\sqrt{2m+1}} E[X_0 1_{|X_0|\le m^{1/s}}] \;=\; O(m^{-1/2}).$$

□

For $d \ge 1$, define $v_d(t) = \big\langle \cos\big(\frac{2\pi j_1 t}{2m+1}\big), \ldots, \cos\big(\frac{2\pi j_d t}{2m+1}\big)\big\rangle$ with respect to distinct integers $1 \le j_1, \ldots, j_d \le m$. Let also $\{N_j\}$ be a sequence of i.i.d. $\mathrm{N}(0,1)$ random variables independent of $\{X_j\}$.

Lemma 5. *For $d \ge 1$, let $\tilde{p}_m$ be the density of*

$$\frac{1}{\sqrt{E[\bar{X}_1^2](2m+1)}}\Big[\sqrt{2}\big(\bar{X}_0 + \sigma_m N_0\big)v_d(0) + 2\sum_{k=1}^m \big(\bar{X}_k + \sigma_m N_k\big)v_d(k)\Big]$$

where $\sigma_m^2 = E[\bar{X}_1^2]s_m^2$. If $m^{-2c_5}\log m \le s_m^2 \le 1$ for $c_5 = 1/2 - (1-\delta)/s > 0$ and some $0 < \delta < 1$, then, uniformly for $|x|^3 = o(m^{1/2-1/s})$,

$$\tilde{p}_m(x) = \phi_{(1+s_m)I_d}(x)(1+o(1)).$$

Proof. Apply Lemma 3 to the centered vectors $\sqrt{2}\bar{X}_0 v_d(0), 2\bar{X}_1 v_d(1), \ldots 2\bar{X}_m v_d(m)$ where, after some calculation,

$$\mathrm{cov}\Big(\sqrt{2}\bar{X}_0 v_d(0) + 2\bar{X}_1 v_d(1) + \cdots + 2\bar{X}_m v_d(m)\Big) \;=\; B_m I_d$$

and

$$B_m \;=\; E\bar{X}_1^2\Big[2 + 4\sum_{k=1}^m \cos^2\big(\frac{2\pi k}{2m+1}\big)\Big] \;=\; (2m+1)E\bar{X}_1^2.$$

Choose for a fixed constant $c_6 > 0$,

$$\tilde{\alpha} \;=\; c_6 m^{-1/s} d^{-1/2}.$$

Note for each $0 \le t \le m$, that

$$|v_d(t)|^2 \;=\; \sum_{l=1}^{d} \cos^2\big(\frac{2\pi j_l t}{2m+1}\big) \;\le\; d.$$

Then, for large m,

$$\begin{aligned}
&\tilde\alpha E|\sqrt{2}\bar X_0 v_d(0)|^3 \exp\{\tilde\alpha\sqrt{2}\bar X_0 v_d(0)\} + \tilde\alpha \sum_{t=1}^{m} E|2\bar X_t v_d(t)|^3 \exp\big\{\tilde\alpha|2\bar X_t v_d(t)|\big\}\\
&\le\; 8d^{3/2}\tilde\alpha(m+1)E|\bar X_1|^3 \exp\big\{2\tilde\alpha|\bar X_1|d^{1/2}\big\}\\
&\le\; 10dc_6 m^{1-1/s} E|\bar X_1|^3 \exp\big\{2c_6\big\}\\
&\le\; 10dc_6 \exp\{4c_6\} m^{1-\delta/s} E|X_1|^{2+\delta}
\end{aligned}$$

where $0 < \delta < 1$ is chosen so that $E|X_1|^{2+\delta} < \infty$. Then, (2) holds with $\alpha = \tilde\alpha$ for sufficiently small c_6.

Now choose

$$\begin{aligned}
\tilde\beta_m \;&=\; B_m^{-3/2} E|\sqrt{2}\bar X_0 v_d(0)|^3 \exp\{\tilde\alpha\sqrt{2}\bar X_0 v_d(0)\}\\
&\quad + B_m^{-3/2} \sum_{t=1}^{m} E|2\bar X_t v_d(t)|^3 \exp\big\{\tilde\alpha|2\bar X_t v_d(t)|\big\}\\
&\le\; 8d^{3/2} B_m^{-3/2}(m+1)E|\bar X_1|^3 \exp\big\{2\tilde\alpha|\bar X_1|d^{1/2}\big\}.
\end{aligned}$$

Then,

$$\tilde\beta_m \;\le\; \mathrm{const}(B_m^{-3/2} m^{1+(1-\delta)/s} E|\bar X_1|^{2+\delta}) \;\le\; \mathrm{const}(m^{-c_5})$$

where $c_5 = 1/2 - (1-\delta)/s > 0$.

Next, we consider (4). We can choose x so that

$$|x| \;\le\; c_1\tilde\alpha B_m^{1/2} \;\sim\; \mathrm{const}(m^{1/2-1/s}).$$

Then, we can choose $\sigma^2 = s_m^2$ so that

$$1 \;\ge\; s_m^2 \;\ge\; \mathrm{const}(m^{-2c_5}\log m)$$

and note

$$B_m \;\sim\; m \;\ge\; c_3\tilde\alpha^{-2} \;\sim\; m^{2/s}.$$

Noting (5), we have

$$\tilde p_m \;=\; \phi_{(1+s_m^2)I_d}(x)\exp\big(\bar T_m(x)\big) \;\;\text{with}\;\; |\bar T_m(x)| \;\le\; c_4\tilde\beta_m(|x|^3+1).$$

However, uniformly over $|x|^3 = o(m^{1/2-1/s})$,

$$|\bar T_m(x)| \;\le\; c_4\tilde\beta_m(|x|^3+1) \;\le\; \mathrm{const}(m^{1/2-1/s-c_5}) \;=\; \mathrm{const}(m^{-\delta/s}) \;=\; o(1).$$

□

Proof of Theorem 1. From properties of circulants, we have that the eigenvalues of (1) are $\lambda_j = X_0 + 2\sum_{k=1}^m \cos\big(\frac{2\pi kj}{2m+1}\big)X_k$ for $0 \le j \le 2m$, and also $\lambda_j = \lambda_{2m+1-j}$ for $1 \le j \le m$. Since $m^{-1/2}\sqrt{2\log m} \to 0$, the variable X_0 in the expression for λ_j can be replaced by $\sqrt{2}\bar X_0$. We will also be able to omit the contribution of λ_0 to the maximum. By Lemma 4, it will be enough to prove

$$\sqrt{2\log m}\Big[\frac{\sqrt{2}\bar X_0}{\sqrt{2m+1}} + \max_{1\le j\le m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^{m} \cos\big(\frac{2\pi jk}{2m+1}\big)\bar X_k - a_m\Big] \;\Rightarrow\; G. \qquad (6)$$

To this end, let $\sigma_m^2 = E[\bar X_1^2] s_m^2 = E[\bar X_1^2] m^{-2c_5} \log m$. We first show

$$\sqrt{2\log m}\Big[\frac{\sqrt{2}(\bar X_0 + \sigma_m N_0)}{\sqrt{E[\bar X_1^2](2m+1)}} \tag{7}$$

$$+ \max_{1\le j\le m} \frac{2}{\sqrt{E[\bar X_1^2](2m+1)}} \sum_{k=1}^m \cos\big(\frac{2\pi jk}{2m+1}\big)(\bar X_k + \sigma_m N_k) - a_m\Big] \ \Rightarrow\ G.$$

For $1 \le j \le m$, let

$$\lambda_j^{\bar X + N} = \frac{1}{\sqrt{E[\bar X_1^2](2m+1)}}\Big[\sqrt{2}(\bar X_0 + \sigma_m N_0) + 2\sum_{k=1}^m \cos\big(\frac{2\pi jk}{2m+1}\big)(\bar X_k + \sigma_m N_k)\Big].$$

Since $1 - e^{-e^{-u}} = \sum_{d=1}^\infty (-1)^{d-1}(e^{-du}/d!)$, by Lemma 2, (7) will follow from the statement

$$P\Big(\lambda_{j_1}^{\bar X+N} > a_m + \frac{u}{\sqrt{2\log m}}, \dots, \lambda_{j_d}^{\bar X+N} > a_m + \frac{u}{\sqrt{2\log m}}\Big)$$
$$= m^{-d}\exp(-du)(1+o(1)) \tag{8}$$

uniformly over the d-tuples $1 \le j_1 < \cdots < j_d \le m$ for each $d \ge 1$ as $m \uparrow \infty$.

Let A_m^d denote the event in the probability on the left-side. Then, noting $s_m^2 = m^{-2c_5}\log m$,

$$\int_{A_m^d} \phi_{(1+s_m^2)I_d}(x)dx \ = \ m^{-d}\exp\{-du\}(1+o(1))$$

as $m \uparrow \infty$. Note that we can neglect the parts in (8) when there is $l \le d$ such that

$$|\lambda_{j_l}^{\bar X+N}|^3 \ > \ m^{1/2-1/s-\epsilon}$$

for a small $\epsilon > 0$. Indeed, given $d \ge 1$ and $s > 2$ choose $0 < \varepsilon < 1/2 - 1/s$ and $\gamma > 2$ such that $\gamma(1/2 - 1/s - \varepsilon) > d+1$. Note also $1/2 \le E[\bar X_1^2] \le 2$ for m large enough. Then, by Rosenthal's inequality there is a constant $C(\gamma)$ such that

$$P\left(\left|\sqrt{2}\bar X_0 + 2\sum_{k=1}^m \cos\big(\frac{2\pi j_l k}{2m+1}\big)\bar X_k\right|^3 > m^{2-1/s-\epsilon}\right)$$
$$\le \ \frac{C(\gamma)}{m^{\gamma(2-1/s-\epsilon)}}\left(\left(\sum_{k=0}^m E|\bar X_k|^2\right)^{3\gamma/2} + mE|\bar X_1|^{3\gamma}\right)$$
$$\le \ C(\gamma)\left(\frac{2}{m^{\gamma(1/2-1/s-\varepsilon)}} + \frac{m}{m^{\gamma(2-4/s-\varepsilon)}}\right) = o(m^{-d}).$$

On the other hand, also

$$P\left(\left|\frac{\sqrt{2}N_0}{\sqrt{2m+1}} + \frac{2}{\sqrt{2m+1}}\sum_{k=1}^m \cos\big(\frac{2\pi j_l k}{2m+1}\big)\sigma_m N_k\right|^3 > m^{1/2-1/s-\varepsilon}\right) = o(m^{-d}).$$

We conclude by Lemma 5 (which does not depend on the choice of $j_1, \dots, j_d$) that (8) holds.

To deduce (6), note $E[\bar{X}_1^2] \to E[X_1^2] = 1$, and

$$\begin{aligned} &\frac{\sqrt{2}\sigma_m N_0}{\sqrt{2m+1}} - \max_{1\le j\le m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\big(\frac{2\pi jk}{2m+1}\big)(-\sigma_m N_k) \\ &\quad \le \frac{\sqrt{2}(\bar{X}_0 + \sigma_m N_0)}{\sqrt{2m+1}} + \max_{1\le j\le m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\big(\frac{2\pi jk}{2m+1}\big)(\bar{X}_k + \sigma_m N_k) \\ &\qquad\quad - \frac{\sqrt{2}\bar{X}_0}{\sqrt{2m+1}} - \max_{1\le j\le m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\big(\frac{2\pi jk}{2m+1}\big)\bar{X}_k \\ &\quad \le \frac{\sqrt{2}\sigma_m N_0}{\sqrt{2m+1}} + \max_{1\le j\le m} \frac{2}{\sqrt{2m+1}} \sum_{k=1}^m \cos\big(\frac{2\pi jk}{2m+1}\big)\sigma_m N_k. \end{aligned} \tag{9}$$

Let $(2m+1)^{1/2}\lambda_j^N = \sqrt{2}\sigma_m N_0 + 2\sum_{k=1}^m \cos\big(2\pi jk/2m+1\big)\sigma_m N_k$ for $1 \le j \le m$. One can calculate that that $\{\lambda_j^N\}_{j=1}^m$ are i.i.d. $\mathrm{N}(0, \sigma_m^2)$ variables.

Hence, to finish, the bounds (9) correspond to the maximum of m i.i.d. $\mathrm{N}(0, \sigma_m^2)$ random variables, well known to be on order $\sigma_m\sqrt{2\log m} \sim m^{-c_5}\log m \to 0$ in probability. $\square$

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