ON THE MAXIMUM CORRELATION COEFFICIENT[*]

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Abstract. For an arbitrary random vector (X, Y) and an independent random variable Z it is shown that the maximum correlation coefficient between X and $Y + \lambda Z$ as a function of λ is lower semicontinuous everywhere and continuous at zero where it attains its maximum. If, moreover, Z is in the class of self-decomposable random variables, then the maximal correlation coefficient is right continuous, nonincreasing for $\lambda \geq 0$ and left continuous, nondecreasing for $\lambda \leq 0$. Independent random variables X and Z are Gaussian if and only if the maximum correlation coefficient between X and $X + \lambda Z$ equals the linear correlation between them. The maximum correlation coefficient between the sum of n arbitrary independent identically distributed random variables and the sum of the first $m < n$ of these equals $\sqrt{m/n}$ (previously proved only for random variables with finite second moments, where it amounts also to the linear correlation). Examples provided reveal counterintuitive behavior of the maximum correlation coefficient for more general Z and in the limit $\lambda \to \infty$.

Key words. dependence, maximum correlation, self-decomposable random variables

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1. Introduction and statement of results. The maximum correlation coefficient between two random elements ξ, η introduced in [6, 5] is

$$(1) \qquad \rho(\xi, \eta) = \sup \left\{ \operatorname{corr}\big(\varphi(\xi), \psi(\eta)\big) \colon\, 0 < \mathbf{E}\big|\varphi(\xi)\big|^2 < \infty,\ 0 < \mathbf{E}\big|\psi(\eta)\big|^2 < \infty \right\},$$

where $\operatorname{corr}(X, Y)$ is the classical (Pearson) correlation between random variables X and Y. Definition (1) is equivalent to

$$(2) \qquad \rho(\xi, \eta) = \sup \mathbf{E}\big\{ \varphi(\xi)\, \psi(\eta) \big\},$$

where the supremum in (2) is taken over all φ, ψ with

$$(3) \qquad \mathbf{E}\varphi(\xi) = \mathbf{E}\psi(\eta) = 0, \quad \mathbf{E}\big|\varphi(\xi)\big|^2 = \mathbf{E}\big|\psi(\eta)\big|^2 = 1.$$

Geometrically, $\rho(\xi, \eta)$ equals the cosine of the angle between the subspaces (of a larger Hilbert space $L^2(\xi, \eta)$), $L^2(\xi) = \{\varphi(\xi)\colon \mathbf{E}\varphi = 0, \mathbf{E}|\varphi|^2 < \infty\}$, and $L^2(\eta) = \{\psi(\eta)\colon \mathbf{E}\psi = 0, \mathbf{E}|\psi|^2 < \infty\}$. Another well-known interpretation of ρ is as the operator norm of the conditional expectation $\varphi \mapsto \mathbf{E}(\varphi(\xi)\,|\,\eta)$ acting on the closed subspace of L_2 consisting of functions orthogonal to constants. Thus

$$(4) \qquad \rho^2(\xi, \eta) = \sup \left\{ \mathbf{E}\left\{ \big|\mathbf{E}\big(\varphi(\xi)|\eta\big)\big|^2 \right\} :\, \mathbf{E}\varphi(\xi) = 0,\ \mathbf{E}\big|\varphi(\xi)\big|^2 = 1 \right\}.$$

The main role of $\rho(\xi, \eta)$ is that of a convenient numerical measure of dependence between ξ and η. In particular, $\rho(\xi, \eta)$ vanishes if and only if ξ and η are independent.

Explicit formulas for $\rho(\xi, \eta)$ are available in very few cases. If (X, Z) is a bivariate Gaussian vector, then

$$(5) \qquad \rho(X, Z) = \big|\operatorname{corr}(X, Z)\big|$$

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(for a proof see, for example, [7]).

If $X = X_1 + \cdots + X_m$, $Z = X_1 + \cdots + X_n$, $m \leqq n$, where $X_1, \ldots, X_n$ are independent identically distributed, nondegenerate random variables with finite second moment, then the maximum correlation between X and Z is

$$\rho(X, Z) = \sqrt{mn^{-1}} \tag{6}$$

(see [4]) and, thus, does not depend on the distribution of X_i. In Corollary 1 below, we show that (6) applies also for any independent identically distributed, nondegenerate random variables X_i. There are a few other isolated cases when $\rho(X, Y)$ is known in an explicit form. Different properties of the maximum correlation were studied in [11, 12, 9, 3, 1].

If X, Y, Z are Markov-dependent, then it is easy to see that

$$\rho(X, Y) \geqq \rho(X, Y + Z). \tag{7}$$

Indeed, since $L^2(Y + Z) \subset L^2(Y, Z)$, one has

$$\rho\big(X, (Y, Z)\big) \geqq \rho(X, Y + Z).$$

From Markov property $\mathbf{E}(\varphi(X) \,|\, Y, Z) = \mathbf{E}(\varphi(X) \,|\, Y)$, which by (4) implies the well-known formula

$$\rho(X, Y) = \rho\big(X, (Y, Z)\big)$$

(cf. the proof of Lemma 1 in [10, p. 207]). Thus (7) follows.

If bivariate random vectors (X_1, Y_1) and (X_2, Y_2) are independent, then

$$\max\big\{\rho(X_1, Y_1), \rho(X_2, Y_2)\big\} \geqq \rho(X_1 + X_2, Y_1 + Y_2). \tag{8}$$

This follows from the fact that $\rho(X_1 + X_2, Y_1 + Y_2) \leqq \rho((X_1, X_2), (Y_1, Y_2))$ and from the Csáki–Fisher identity

$$\rho\big((X_1, X_2), (Y_1, Y_2)\big) = \max\big\{\rho(X_1, Y_1), \rho(X_2, Y_2)\big\};$$

see Theorem 1 in [13]. Inequality (8) yields two implications in the same spirit as (7).

(i) If (X', Y') is an independent copy of (X, Y), then

$$\rho(X, Y) \geqq \rho(X + X', Y + Y'). \tag{9}$$

(ii) If (X, Y) is an arbitrary bivariate random vector and Z_1, Z_2 are independent of each other and of (X, Y), then

$$\rho(X, Y) \geqq \rho(X + Z_1, Y + Z_2).$$

We prove in this paper the following general properties of $\rho(X, Y + \lambda Z)$.

THEOREM 1. *The function $\lambda \mapsto \rho(X, Y + \lambda Z)$ is lower semicontinuous in λ for any random variables X, Y, Z. In particular, if Z is independent of the pair (X, Y), then $\rho(X, Y + \lambda Z)$ is continuous at $\lambda = 0$.*

A random variable Z is in $\mathcal{L}$ if for any c, $0 < c < 1$, there exists a random variable U_c independent of Z such that

$$Z \quad \text{is equidistributed with} \quad cZ + U_c. \tag{10}$$

Equivalently, a real-valued random variable Z belongs to the class $\mathcal{L}$ if its characteristic function $f(t) = \mathbf{E}e^{itZ}$, $t \in \mathbf{R}$, possesses the following property: for any c, $0 < c < 1$, there exists a characteristic function $f_c(t)$ such that

$$f(t) = f(ct)\, f_c(t), \qquad t \in \mathbf{R}. \tag{11}$$

The random variables in $\mathcal{L}$ are called self-decomposable. All random variables in $\mathcal{L}$ are infinitely divisible. Necessary and sufficient conditions (in terms of Lévy functions) are

known for an infinitely divisible random variable to belong to $\mathcal{L}$ (see [8, Chap. 5]). In particular, all random variables having stable distributions are in $\mathcal{L}$.

We next detail additional properties of $\rho(X, Y + \lambda Z)$ in case when Z is independent of (X, Y) and belongs to the class $\mathcal{L}$.

THEOREM 2. *If a random variable $Z \in \mathcal{L}$ is independent of a bivariate random vector (X, Y), then $\lambda \mapsto \rho(X, Y + \lambda Z)$ is a nonincreasing right continuous function on $[0, \infty)$ and a nondecreasing left continuous function on $(-\infty, 0]$.*

The above results hold for random elements X, Y, Z taking values in an arbitrary separable Banach space; the proofs remain the same.

The following theorem is a converse of (5) holding when X and Z are Gaussian.

THEOREM 3. *If X and Z are independent, nondegenerate, square-integrable real-valued random variables such that for every real λ*

$$(12) \qquad \rho(X, X + \lambda Z) = \operatorname{corr}(X, X + \lambda Z),$$

then X and Z are Gaussian.

Dembo, Kagan, and Shepp [4] show that equality (12) may hold true for a fixed $\lambda \neq 0$ with independent, nondegenerate square-integrable *non-Gaussian* X and Z; see (6).

Our following result provides the value of $\rho(X, X + \lambda Z)$ in case both X and Z are symmetric α-stable random variables.

THEOREM 4. *Suppose X and Z are independent copies of an α-stable random variable $\alpha \in (0, 2]$. Then*

$$(13) \qquad \rho(X, X + \lambda Z) = \frac{1}{\sqrt{1 + |\lambda|^{\alpha}}}$$

for all $\lambda \geqq 0$. If X and Z are symmetric, equality (13) holds also for $\lambda < 0$.

The following lemma, which is key to the proof of Theorem 4, is of independent interest.

LEMMA 1. *Suppose X and Y are nondegenerate independent random variables with characteristic functions $\varphi_X(t)$ and $\varphi_Y(t)$ such that*

$$(14) \qquad \liminf_{t \to 0} \frac{1 - |\varphi_Y(t)|^2}{1 - |\varphi_X(t)|^2} = c.$$

Then

$$(15) \qquad \rho(X, X + Y) \geqq \frac{1}{\sqrt{1 + c}}.$$

Suppose X_j are independent identically distributed, nondegenerate random variables with characteristic function $\varphi(t)$ (possibly with infinite second moment). The independent random variables $X = \sum_{j=1}^{m} X_j$ and $Y = \sum_{j=m+1}^{n} X_j$ have characteristic functions $\varphi_X(t) = \varphi^m(t)$ and $\varphi_Y(t) = \varphi^{n-m}(t)$. Applying Lemma 1 for the pair (X, Y), where $c = (n - m)/m$ by the continuity of $|\varphi(t)|^2$ at $t = 0$, we get that

$$(16) \qquad \rho(X, X + Y) \geqq \sqrt{mn^{-1}}.$$

Combining this lower bound with the upper bound of inequality (19) of [4], we get the following corollary.

COROLLARY 1. *Equality (6) holds for any nondegenerate, independent identically distributed $X_1, \ldots, X_n$.*

2. Proofs.

Proof of Theorem 1. Note that if $\psi_n \to \psi$ in L^2 and $\varphi_n \to \varphi$ in L^2, with both φ and ψ nonzero (in L^2), then $\operatorname{corr}(\varphi_n, \psi_n) \to \operatorname{corr}(\varphi, \psi)$. Consequently, in definition (1) it suffices to consider φ and ψ in the dense subsets BL of $L^2(\xi)$ and $L^2(\eta)$, consisting of bounded Lipschitz functions, using the notation $\|\psi\|_{BL} = \|\psi\|_\infty + \|\psi\|_{\text{Lip}}$. Fix φ, ψ bounded and Lipschitz having positive variances at X and $Y + tZ$, respectively. Define $\Delta(r) = \min(1, |r|\|Z\|)$, so

$$\left|\psi(Y + tZ) - \psi(Y + sZ)\right| \leqq 2\|\psi\|_{BL}\Delta(t - s),$$

implying that

$$\left|\operatorname{cov}\big(\varphi, \psi(Y + tZ)\big) - \operatorname{cov}\big(\varphi, \psi(Y + sZ)\big)\right| \leqq 4\|\varphi\|_\infty \|\psi\|_{BL}\mathbf{E}\big(\Delta(t - s)\big),$$

$$\left|\mathbf{D}\big(\psi(Y + tZ)\big) - \mathbf{D}\big(\psi(Y + sZ)\big)\right| \leqq 8\|\psi\|_{BL}^2\mathbf{E}\big(\Delta(t - s)\big).$$

By the bounded convergence $\mathbf{E}(\Delta(t - s)) \to 0$ as $s \to t$, implying that $\mathbf{D}\{\psi(Y + sZ)\}$ is bounded away from 0 in a neighborhood of t, so also

$$\operatorname{corr}\big\{\varphi, \psi(Y + sZ)\big\} \to \operatorname{corr}\big\{\varphi, \psi(Y + tZ)\big\} \qquad \text{as} \quad s \to t.$$

Thus

$$\lambda \longmapsto \sup\Big\{\operatorname{corr}\big(\varphi, \psi(Y + \lambda Z)\big)\colon \mathbf{D}\big(\varphi(X)\big),\ \mathbf{D}\big(\psi(Y + \lambda Z)\big) \text{ are finite}\Big\}$$

is lower semicontinuous. This ends the proof of the first part of the theorem.

Combining the first part of the theorem and inequality (7) for λZ, we have

$$\rho(X, Y) \geqq \liminf_{\lambda \to 0} \rho(X, Y + \lambda Z) \geqq \rho(X, Y),$$

proving the continuity of $\rho(X, Y + \lambda Z)$ at $\lambda = 0$ and completing the proof of Theorem 1.

Proof of Theorem 2. Let $\lambda_2 > \lambda_1 > 0$; write $\lambda_1 = c\lambda_2$, where $0 < c < 1$. Let U_c be a random variable independent of X, Y, Z such that $Z \cong cZ + U_c$. From (7),

$$\rho(X, Y + \lambda_1 Z) = \rho(X, Y + \lambda_2 cZ) \geqq \rho(X, Y + \lambda_2 cZ + \lambda_2 U_c)$$
$$= \rho\big(X, Y + \lambda_2(cZ + U_c)\big) = \rho(X, Y + \lambda_2 Z).$$

If $\lambda_2 < \lambda_1 < 0$, then setting $\lambda_2' = -\lambda_2,\ \lambda_1' = -\lambda_1$ one has

$$\rho(X, Y + \lambda_1 Z) = \rho(-X, -Y - \lambda_1 Z) = \rho(-X, -Y + \lambda_1' Z)$$
$$\geqq \rho(-X, -Y + \lambda_2' Z) = \rho(X, Y - \lambda_2' Z) = \rho(X, Y + \lambda_2 Z).$$

Inequality (7), applied for λZ, extends the above monotonicity properties of $\lambda \mapsto \rho(X, Y + \lambda Z)$ to $[0, \infty)$ and $(-\infty, 0]$, respectively. By Theorem 1 this function is lower semicontinuous in λ; hence it is right continuous wherever nonincreasing, and left continuous wherever nondecreasing.

Proof of Theorem 3. We assume without loss of generality that $\mathbf{E}X = \mathbf{E}Z = 0$, $\mathbf{E}X^2 = \mathbf{E}Z^2 = 1$. Using (12) with $\lambda = 1/s > 0$ we get $\rho(X, Z + sX) = \operatorname{corr}(X, Z + sX)$, and hence

$$(17) \qquad\qquad \mathbf{E}(X \mid Z + sX) = \frac{s}{1 + s^2}(Z + sX)$$

(see [4, p. 344]). Replacing Z by $-Z$ in (12), it is easy to see that (17) holds also for $s \leqq 0$. This implies

$$\mathbf{E}\big[X \exp\big\{it(Z + sX)\big\}\big] = \frac{s}{1 + s^2}\mathbf{E}\big[(Z + sX)\exp\big\{it(Z + sX)\big\}\big].$$

Differentiating this relation with respect to s at $s = 0$ we get

$$it\mathbf{E}\big\{X^2 \exp(itZ)\big\} = \mathbf{E}\big\{Z \exp(itZ)\big\}.$$

Since X, Z are independent and $\mathbf{E}X^2 = 1$, this shows that the characteristic function $\varphi(t) = \mathbf{E}\exp(itZ)$ satisfies the differential equation $\varphi'(t) = -t\varphi(t)$, and hence Z is Gaussian. With $u = 1/s$, it follows from (17) that

$$\mathbf{E}(Z \mid X + uZ) = \mathbf{E}(Z \mid Z + sX) = (Z + sX) - s\,\mathbf{E}(X \mid Z + sX) = \frac{u}{1 + u^2}\,(X + uZ).$$

Reversing the roles of X and Z, by the same argument as before X is also Gaussian.

Proof of Lemma 1.   Recall that for characteristic function the $|\varphi(t)|^2 \leqq 1$ holds. So, fixing $t \in \mathbf{R}$ such that $|\varphi_X(t)| \neq 1$, and considering separately the real and imaginary parts of $f(x) = e^{itx}$, it is easy to check that (4) implies

$$\rho^2(X, X + Y) \geqq \frac{\mathbf{E}|\mathbf{E}(f(X + Y) \mid X)|^2 - |\mathbf{E}f(X + Y)|^2}{\mathbf{E}|f(X + Y)|^2 - |\mathbf{E}f(X + Y)|^2}.$$

We have $|\mathbf{E}f(X + Y)|^2 = |\varphi_X(t)\varphi_Y(t)|^2$,

$$\left|\mathbf{E}\big(f(X + Y) \mid X\big)\right|^2 = \big|\varphi_Y(t)\big|^2, \qquad \big|f(X + Y)\big|^2 = 1.$$

Thus, if in addition $|\varphi_Y(t)| \neq 0$,

$$\rho^2(X, X + Y) \geqq \frac{|\varphi_Y(t)|^2(1 - |\varphi_X(t)|^2)}{1 - |\varphi_X(t)|^2|\varphi_Y(t)|^2} = \frac{1}{1 + (|\varphi_Y(t)|^{-2} - 1)/(1 - |\varphi_X(t)|^2)}.$$

Taking now the lim sup of the right-hand side as $t \to 0$, we get the conclusion (15) out of our assumption (14).

Proof of Theorem 4.   Applying Lemma 1 to $Y = \lambda Z$ we get that

$$(18) \qquad\qquad\qquad \rho(X, X + \lambda Z) \geqq \frac{1}{\sqrt{1 + |\lambda|^\alpha}}.$$

If X and Z are symmetric, the pairs $(X, X + \lambda Z)$ and $(X, X - \lambda Z)$ have the same distribution. Hence, it suffices to prove the converse of (18) for $\lambda > 0$. To this end, fix $0 < \varepsilon < \lambda$ and let $m < n$ be positive integers such that $\lambda - \varepsilon < (n/m - 1)^{1/\alpha} < \lambda$. Then, by Theorem 2 (and by the invariance of ρ under nondegenerate linear transformations),

$$\rho(X, X + \lambda Z) \leqq \rho(X, X + (nm^{-1} - 1)^{1/\alpha}Z)$$
$$= \rho\big(m^{1/\alpha}X, m^{1/\alpha}X + (n - m)^{1/\alpha}Z\big) = \rho(S_m, S_n),$$

where S_n denotes the sum of n independent copies of the α-stable random variable X. Therefore, inequality (19) in [4] gives

$$\rho(X, X + \lambda Z) \leqq \sqrt{\frac{m}{n}} \leqq \frac{1}{\sqrt{1 + (\lambda - \varepsilon)^\alpha}}\ .$$

Since $\varepsilon > 0$ is arbitrary, this ends the proof.

3. Some counterintuitive examples. Here some examples are constructed that demonstrate counterintuitive features of the maximum correlation.

Example 1.   Let Z be a nondegenerate random variable independent of (X, Y). As $|\lambda| \to \infty$, one may ask when X and $Y + \lambda Z$ become "asymptotically independent," i.e.,

$$(19) \qquad\qquad\qquad \lim_{|\lambda|\to\infty} \rho(X, Y + \lambda Z) = 0.$$

From [2] it follows that for bounded φ, ψ

$$\lim_{|\lambda|\to\infty} \text{cov}\big\{\varphi(X), \psi(Y + \lambda Z)\big\} = 0$$

if Z has a density, and that

$$\lim_{|\lambda| \to \infty} \mathrm{cov}\big\{\varphi(X), \psi(Y + \lambda Z)\big\} = \mathrm{cov}\big\{\varphi(X), \psi(Y)\big\}$$

if Z is discrete. The latter shows that (19) does not hold in general. Here is a related explicit example. Let X be an arbitrary (nondegenerate) random variable, let the distribution of Y be concentrated on $[-\frac{1}{2}, \frac{1}{2}]$, and let Z be a binary random variable taking values -1 and $+1$. For any (known) λ with $|\lambda| > 1$, Y can be reconstructed from $Y + \lambda Z$ implying that for $|\lambda| > 1$, $\rho(X, Y + \lambda Z) = \rho(X, Y)$ and does not go to 0 as $|\lambda| \to \infty$.

Example 2.   These X, Y, Z would give an example of $\rho(X, Y + \lambda Z)$ that does not decrease monotonically in $\lambda \in (0, \infty)$ unless $\rho(X, Y + \lambda Z) \equiv \rho(X, Y)$.

We shall now show that if $X = Y$ with $\mathbf{P}\{X = -\frac{1}{2}\} = \mathbf{P}\{X = \frac{1}{2}\} = \frac{1}{2}$ and Z is independent of X with $\mathbf{P}\{Z = -1\} = \mathbf{P}\{Z = 1\} = \frac{1}{2}$, then $\rho(X, X + \lambda Z) = 1/\sqrt{2} < 1$ for $\lambda = \frac{1}{2}$.

The random variable $X + Z/2$ takes values $-1, 0, +1$ with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, respectively. In view of (3), one may always assume

$$\varphi\left(-\frac{1}{2}\right) = -1, \qquad \varphi\left(\frac{1}{2}\right) = 1.$$

Then $\mathbf{E}(\varphi(X)|X + Z/2 = \pm 1) = \pm 1$ and $\mathbf{E}(\varphi(X)|X + Z/2 = 0) = 0$. Thus $\rho^2 = \sup_\varphi \mathbf{E}\{[\mathbf{E}(\varphi(X)|X + Z/2)]^2\} = \frac{1}{2}$ by (4).

Example 3.   The random variables X, Z from Example 2 are also an example of $\rho(X, X + \lambda Z)$ which, as a function of λ, is discontinuous at $\lambda = \frac{1}{2}$ and $-\frac{1}{2}$. Indeed, for any λ with $|\lambda| \neq \frac{1}{2}$, X can be reconstructed from $X + \lambda Z$, whence

$$\rho(X, X + \lambda Z) = 1, \qquad |\lambda| \neq \frac{1}{2},$$

while $\rho(X, X \pm Z/2) < 1$ as shown above.

This construction can easily be generalized to X and Z with finite > 2 number of values such that the continuity of $\rho(X, X + \lambda Z)$ fails to hold at prescribed $\lambda_i > 0$, $i = 1, \ldots, k$.

Taking X, Z as above and Y concentrated on $[-\frac{1}{6}, \frac{1}{6}]$, independent of Z, such that $1 > \rho(X, Y) > 1/\sqrt{2}$, we now see by Example 1 that for $\lambda = \frac{1}{2}$,

$$\rho(X, Y + \lambda Z) = \rho(X, Y) > \rho(X, X + \lambda Z),$$

the dependence between X and $Y + \lambda Z$ is stronger than the dependence between X and $X + \lambda Z$ at $\lambda = \frac{1}{2}$, whereas the opposite relationship holds at $\lambda = 0$.

Example 4.   The asymptotic independence (19) may fail even when $X = Y$ and Z are both in $\mathcal{L}$. Indeed, let X be an α-stable random variable and let Z be a β-stable random variable independent of X for some $0 < \alpha < \beta \leqq 2$ ($\beta = 2$ in the case where Z is normal). With (X', Z') denoting an independent copy of (X, Z), the distribution of $(X + X', Z + Z')$ equals up to a nonrandom constant, that of $(2^{1/\alpha}X, 2^{1/\beta}Z)$. Hence, by (9),

$$\rho(X, X + \lambda Z) \geqq \rho\big(X + X', X + X' + \lambda(Z + Z')\big) = \rho(X, X + \lambda 2^{1/\beta - 1/\alpha}Z).$$

Since $2^{1/\beta - 1/\alpha} < 1$, Theorem 2 provides the reverse inequality, implying that $\rho(X, X + \lambda Z)$ is constant on $(0, \infty)$ and constant on $(-\infty, 0)$. By Theorem 1 this function of λ is continuous at $\lambda = 0$; hence $\rho(X, X + \lambda Z) = \rho(X, X) = 1$. Obviously, (19) fails to hold in this case.

4. Open problems. I. As shown in Theorem 2, $Z \in \mathcal{L}$ implies monotonicity of $\rho(X, Y + \lambda Z)$ in λ for any (X, Y) independent of Z.

It is interesting to investigate whether the condition $Z \in \mathcal{L}$ is sufficient for continuity of $\rho(X, Y + \lambda Z)$ at $\lambda \neq 0$. More generally, what is the class of Z for which monotonicity or continuity of $\lambda \mapsto \rho(X, Y + \lambda Z)$ applies for *all* (X, Y)?

II. More generally, one may consider the properties of

$$\rho_{p,q}(\xi,\eta) = \sup \left\{ \frac{\mathrm{cov}(U,V)}{\|U\|_p\|V\|_q} : \ U \in L^p(\xi), V \in L^q(\eta), U, V \neq 0 \right\}$$

for $1/p + 1/q \leqq 1$, where $\rho = \rho_{2,2}$. Of particular interest are $\rho_{\infty,\infty}$ corresponding to strong mixing, and $\rho_{\infty,1}$ and $\rho_{1,\infty}$ corresponding to uniform strong mixing.

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