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Conditional expectation with respect to dependent σ-fields

In this note we will deal with random variables (X_k), which are the conditional expectation of some random variable X with respect to σ-fields $(\mathbf{F}_k)$. In the case where $(\mathbf{F}_k)$ are independent σ-fields such random variables were characterized by Bryc and Kwapień [2] and by a similar technique part of their results was carried over to some dependent σ-fields by Bryc [3]. The asymptotic behaviour of (X_k) for Markov dependent σ-fields $(\mathbf{F}_k)$ was studied by Isaac [4].

Here we will concentrate on the case of two σ-fields $\mathbf{F}_1$, $\mathbf{F}_2$ and we will look for the conditions close to being necessary for all reasonable X_1, X_2 to be of the form $X_k = E^{\mathbf{F}_k}(X)$, $k = 1, 2$; cf. the theorem below.

We assume in the sequel that each $\mathbf{F}_k$, $k = 1, 2$, is a P-complete σ-field. Let $\mathbf{G} = \mathbf{F}_1 \cap \mathbf{F}_2$ and denote by $L_p^0(\mathbf{F}_k)$ the Banach subspace of L_p, $1 \leqslant p \leqslant \infty$, generated by $\mathbf{F}_k$-measurable random variables Y such that $E^{\mathbf{G}}(Y) = 0$, $k = 1, 2$ (L_p denotes the Banach space of all p-integrable random variable with the usual norm). Let us observe that the conjugated space to $L_p^0(\mathbf{F}_k)$ is isometrically isomorphic to $L_{p'}^0(\mathbf{F}_k)$, where $p < \infty$, p and p' are conjugated numbers, i.e. $1/p + 1/p' = 1$ ($\infty' = 1$).

First, let us observe that if there exists a random variable $X \in L_1$ such that $X_k = E^{\mathbf{F}_k}(X)$, $k = 1, 2$, then $E^{\mathbf{G}}X_1 = E^{\mathbf{G}}X_2$. Moreover it is possible to assume that $E^{\mathbf{G}}X_k = 0$, $k = 1, 2$, since evidently the assertions

(i) $\forall\ X_k \in L_p(\mathbf{F}_k)$, $E^{\mathbf{G}}(X_1) = E^{\mathbf{G}}(X_2) \Rightarrow \exists\ Z \in L_p$ such that $X_k = E^{\mathbf{F}_k}(Z)$, $k = 1, 2$,

and

(ii) $\forall\ X_k \in L_p^0(\mathbf{F}_k)$ $\exists\ Z \in L_p$ such that $X_k = E^{\mathbf{F}_k}(Z)$, $k = 1, 2$,

are equivalent. This observation motivates the introduction of the spaces L_p^0.

Lemma 1. *Let $1 \leqslant p \leqslant \infty$. Then the following conditions are equivalent*

(A) *For every $X_k \in L_p^0(\mathbf{F}_k)$, $k = 1, 2$, there exists $Z \in L_p$ such that $X_k = E^{\mathbf{F}_k}(Z)$, $k = 1, 2$;*

(B) *There exists $C > 0$ such that for every $Y_k \in L_{p'}^0(\mathbf{F}_k)$*

$$\|Y_1\|_{p'} + \|Y_2\|_{p'} \leqslant C\, \|Y_1 + Y_2\|_{p'}.$$

Proof. We will prove the lemma only for $1 \leqslant p < \infty$. If $p = \infty$ some changes are needed in the reasoning (a conjugate version of the Banach theorem has to be used). Let $T\colon L_p \to L_p^0(\mathbf{F}_1) \times L_p^0(\mathbf{F}_2)$ be defined by $T(X) = ((E^{\mathbf{F}_k} - E^{\mathbf{G}})(X))_{k=1,2}$. Then $T^*\colon L_{p'}^0(\mathbf{F}_1) \times L_{p'}^0(\mathbf{F}_2) \to L_{p'}$ is easily seen to be defined by $T^*((Y_1, Y_2)) = Y_1 + Y_2$. By the Banach theorem (see, e.g., Rudin [6], Th. 4.15) the linear operator T is "onto" if the adjoint operator

T^* is an isomorphic embedding, i.e. there exists a constant C such that $|||(Y_1, Y_2)||| \leqslant C \| T^*((Y_1, Y_2)) \|_{p'}$, where $|||\cdot|||$ is any of the equivalent norms on $L^0_{p'}(\mathbf{F}_1) \times L^0_{p'}(\mathbf{F}_2)$. Choosing $|||(Y_1, Y_2)||| = \| Y_1 \|_{p'} + \| Y_2 \|_{p'}$ we conclude the proof.

Let $\rho = \sup\{E(X_1 X_2) : X_k \in L^0_2(\mathbf{F}_k), \| X_k \|_2 = 1, k = 1, 2\}$. Note that $0 \leqslant \rho \leqslant 1$. In the case where $\mathbf{G}$ is the trivial σ-field ρ is known as the *maximal correlation coefficient*. The rôle of ρ in the context of conditional expectations is partially explained by

Corollary 1. *Let $\mathbf{F}_1$, $\mathbf{F}_2$ be σ-fields such that for every $X_k \in L^0_2(\mathbf{F}_k)$, $k = 1, 2$, there exists $Z \in L_2$ such that $X_k = E^{\mathbf{F}_k}(Z)$, $k = 1, 2$. Then $\rho < 1$.*

Proof. On account of Lemma 1 the corollary follows from the inequality $E\,|X_1 + X_2|^2 \leqslant (1 - \rho)\,(E\,|X_1|^2 + E\,|X_2|^2)$.

Note that since the inequality in the proof of Corollary 1 can be arbitrarily close to equality the converse implication holds, too. This will be reproved in more constructive a way in our main result

Theorem. *Let $1 < p < \infty$ and assume $\mathbf{F}_1$, $\mathbf{F}_2$ are such that $\rho < 1$. Then for every $X_1 \in L^0_p(\mathbf{F}_1)$, $X_2 \in L^0_p(\mathbf{F}_2)$ there exists $Z \in L_p$ such that $X_k = E^{\mathbf{F}_k}(Z)$, $k = 1, 2$.*

As an immediate consequence of the theorem we have

Corollary 2. *If $\mathbf{F}_1$, $\mathbf{F}_2$ are σ-fields such that for every $X_k \in L^0_p(F_k)$ there exists $Z \in L_2$ such that $X_k = E^{\mathbf{F}_k}(Z)$, $k = 1, 2$, then for every $X_k \in L^0_p(\mathbf{F}_k)$ there exists $Z \in L_p$ such that $X_k = E^{\mathbf{F}_k}(Z)$, $k = 1, 2$, $\forall\ 1 < p < \infty$.*

For the proof of the theorem we will need one more lemma which strengthens the convergence in the "alterniende Verfahren" (Rota [5]).

Lemma 2. *Let $1 < p < \infty$, $\rho < 1$. If $E^{\mathbf{F}_1}E^{\mathbf{F}_2}$ is considered as a linear operator acting on L_p, then $(E^{\mathbf{F}_1}E^{\mathbf{F}_2})^k \to E^{\mathbf{G}}$ as $k \to \infty$ in the norm topology. The same holds for $E^{\mathbf{F}_2}E^{\mathbf{F}_1}$.*

Proof. Let $Q = E^{\mathbf{F}_1}E^{\mathbf{F}_2}$. First, let us observe that if Q is considered as acting on L^0_2, then $\| Q \| \leqslant \rho$. Indeed, $\| Qf \|^2 = E(E^{\mathbf{F}_1}E^{\mathbf{F}_2}(f)\, E^{\mathbf{F}_1}E^{\mathbf{F}_2}(f)) = E(E^{\mathbf{F}_2}(f)\, E^{\mathbf{F}_1}E^{\mathbf{F}_2}(f)) \leqslant \rho \| f \|_2 \| Qf \|_2$. Therefore $Q^k \to 0$ as $k \to \infty$ when Q is considered as an operator acting on L^0_2, and since $\| Q^k - E^{\mathbf{G}} \|_{L_2 \to L_2} = \| Q^k(I - E^{\mathbf{G}}) \|_{L_2 \to L_2} \leqslant 2 \| Q^k \|_{L^0_2 \to L^0_2} \to 0$, the lemma is proved for $p = 2$.

Now, let $1 < p < 2$. Then, since $\| Q^k - E^{\mathbf{G}} \|_{L_1 \to L_1} \leqslant 2$, by the Riesz convexity theorem ([1], Th. 1.1.1.) $\| Q^k - E^{\mathbf{G}} \|_{L_p \to L_p} \leqslant C \| Q^k - E^{\mathbf{G}} \|^{2(p-1)/p}_{L_2 \to L_2} \to 0$ as $k \to \infty$.

A similar argument with L_∞ instead of L_1 proves the lemma in the case $2 < p < \infty$.

By symmetry the same holds for $E^{\mathbf{F}_2}E^{\mathbf{F}_1}$.

Proof of the theorem. By Lemma 2 both series $\sum_{k=0}^{\infty} (E^{\mathbf{F}_1}E^{\mathbf{F}_2})^k$ and $\sum_{k=0}^{\infty} (E^{\mathbf{F}_2}E^{\mathbf{F}_1})^k$ are convergent in the operator norm, when considered as linear operators acting on the Banach space L^0_p. Indeed, $(E^{\mathbf{F}_1}E^{\mathbf{F}_2})^k = (E^{\mathbf{F}_1}E^{\mathbf{F}_2})^k - E^{\mathbf{G}}$ on L^0_p, hence $\| (E^{\mathbf{F}_1}E^{\mathbf{F}_2})^{n_0} \| < 1$ for some n_0 large enough and $\| (E^{\mathbf{F}_1}E^{\mathbf{F}_2})^{kn_0} \| \leqslant \| (E^{\mathbf{F}_1}E^{\mathbf{F}_2})^{n_0} \|^k$.

To complete the proof it suffices now to notice that if $X_k \in L^0_p(\mathbf{F}_k)$, $k = 1, 2$, then $X_1 - E^{\mathbf{F}_1}(X_2)$ and $X_2 - E^{\mathbf{F}_2}(X_1)$ are in L^0_p. Therefore

$$Z = \sum_{k=0}^{\infty} (E^{\mathbf{F}_1}E^{\mathbf{F}_2})^k (X_1 - E^{\mathbf{F}_1}(X_2)) + \sum_{k=0}^{\infty} (E^{\mathbf{F}_2}E^{\mathbf{F}_1})^k (X_2 - E^{\mathbf{F}_2}(X_1))$$

is a well-defined p-integrable random variable. A direct computation shows that $E^{F_k}(Z) = X_k$, $k = 1, 2$.

Remark. During the Braşov Conference the author was convinced that the theorem holds at least for $p=1$ without any assumptions on the dependence of σ-fields. This appeared not to be true. To see this let us consider the interval $[0, 1]$ with the uniform probability distribution and the following σ-fields: $\mathbf{F}_1$ is generated by a family I of intervals such that $I = \bigcup_{n=0}^{\infty} I_n$ where I_n is any finite partition of $(2^{-n-1}, 2^{-n})$ consisting of intervals of length less than $n^{-1}2^{-n-1}$; $\mathbf{F}_2$ is generated by a family J of intervals such that $J = \bigcup_{n=0}^{\infty} J_n$, where J_n is a finite partition of $(2^{-n-1}, 2^{-n})$ generated by the centers of intervals in I_n (thus the intervals in J_n are of length less than $n^{-1}2^{-n-1}$, too). Then the σ-field $\mathbf{G} = \mathbf{F}_1 \cap \mathbf{F}_2$ is generated by the intervals $(2^{-n-1}, 2^{-n})$. To prove that there are $\mathbf{F}_k$-measurable random variables X_k, $k = 1, 2$, such that $E^{\mathbf{G}}(X_1) = E^{\mathbf{G}}(X_2)$ and which cannot be of the form $X_k = E^{\mathbf{F}_k}(Z)$, $k = 1, 2$, for any integrable Z, we use Lemma 1. Namely, we will show that there is no constant $C > 0$ such that for every $Y_1 \in L^0_\infty(\mathbf{F}_1)$, $Y_2 \in L^0_\infty(\mathbf{F}_2)$

$$\|Y_1\|_\infty + \|Y_2\|_\infty \leqslant C\,\|Y_1 - Y_2\|_\infty \tag{1}$$

To this end let $n \in N$ be fixed and define

$$Y(\omega) = \begin{cases} 2^{n+1}\,\omega - 1 & \text{if } \ 2^{-n-1} < \omega < 2^{-n} \\ 0 & \text{otherwise} \end{cases}$$

and

$$Y_k = E^{\mathbf{F}_k}(Y),\ k = 1, 2.$$

Then, since J_n consists of intervals of length less than $n^{-1}2^{-n-1}$ and Y is a Lipschitz function with constant 2^{n+1}, we have $\|Y - Y_2\|_\infty < n^{-1}$ and, similarly, $\|Y - Y_1\|_\infty < n^{-1}$. Thus, $\|Y_1 - Y_2\|_\infty < 2n^{-1}$, which contradicts (1) since n is arbitrary and $\|Y_1\|_\infty, \|Y_2\|_\infty \geqslant (n-1)/n$.

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