TORIC CODE (Kitaev 1997)

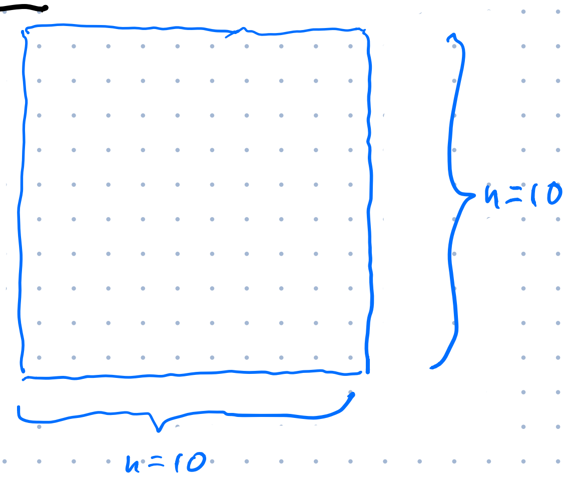
- Encode 2 qubits in N qubits, $N = 2n^2$

  $\dim(\mathcal{H}_c) = 2^2 \qquad \dim(\mathcal{H}) = 2^N$

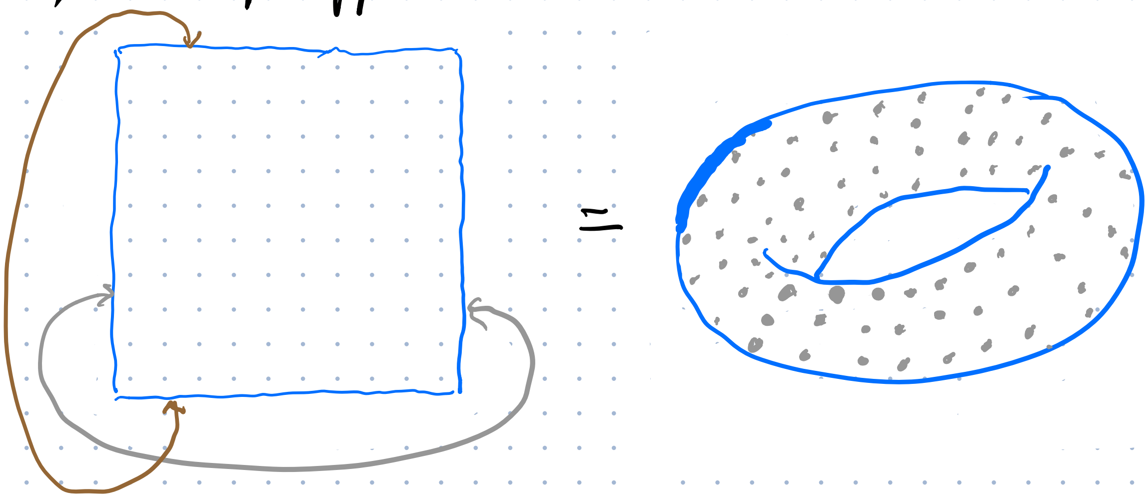
- Corrects against $n \sim \sqrt{N}$ 1-qubit errors
- Has highest-known error threshold
- Conceptually simple
- Leads to notion of topological quantum computing
  = build quantum computer out of a material which automatically (naturally) error-corrects!

To describe, do not think of quantum circuit diagram, but rather of an arrangement of qubits in space:

o) Imagine an $n \times n$ square lattice of points = "vertices"

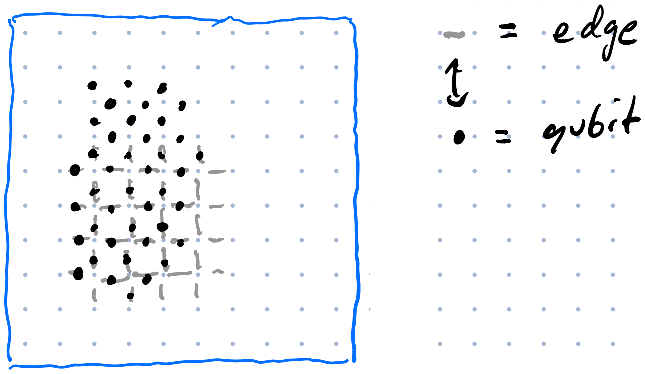


•) Identify opposite sides of square:



= surface of donut = "torus"

•) put 1 qubit on each edge of lattice



$\Rightarrow$ total no. qubits $= 2n^2$.

From now on, we just write lattice with understanding that qubits are located on the edges.

Label each edge by an index $\ell \in \{1, \dots, 2n^2\}$

So the ℓ^{th} qubit Hilbert space has basis

$\mathcal{H}_\ell \simeq \{ |0\rangle_\ell , |1\rangle_\ell \}$ which we notate:

$\updownarrow$ $\quad$ $\updownarrow$

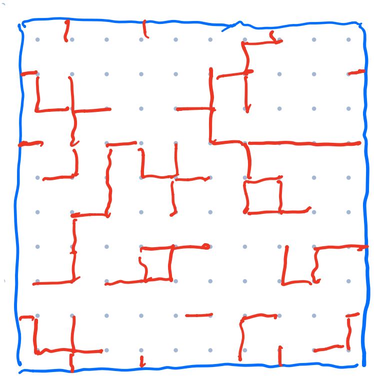
color edge red for $|1\rangle$

ℓ^{th} edge $\quad$ ℓ^{th} edge

don't color edge for $|0\rangle$

A general computational basis state of

$\mathcal{H} = \bigotimes_{\ell=1}^{2n^2} \mathcal{H}_\ell$ is:

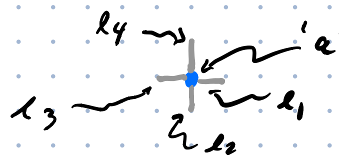


i.e. a set of (0's or 1's) on each edge

- Now want to define a set of error syndrome operators $F_n = \{ V_a , P_{\hat{a}} \}$

- Vertex operators V_a

Label the vertices by $a, b, .. \in \{1, ..., n^2\}$
Write "$\ell \in a$" to mean edge "ℓ" ends on vertex "a"



$\ell_i \in a$

Then

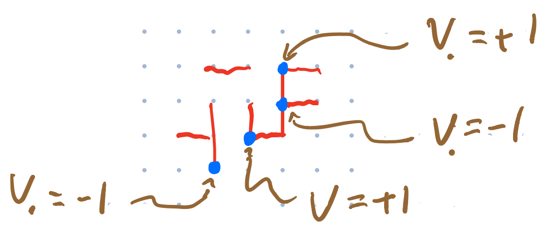
$$\boxed{V_a \doteq \prod_{l\in a} Z_l = Z_{l_1} Z_{l_2} Z_{l_3} Z_{l_4}}$$

$Z_l =$ Z-gate acting on l^{th} qubit

$$\Rightarrow \begin{cases} V_a V_b = V_b V_a \\ V_a^2 = I \\ \text{eigenvalues } V_a \in \{\pm 1\} \end{cases}$$

$+1$ eigenstates = even number of red edges $\in a$

-1 " = odd " " " " $\in a$

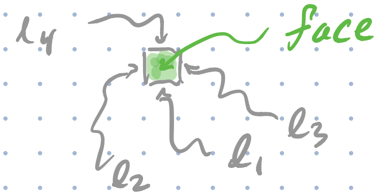


$$\Rightarrow \boxed{\prod_{a=1}^{n^2} V_a = I}$$

since each Z_l appears twice. $\Rightarrow$ n^2-1 independent $\{V_a\}$ ops.

- <u>Plaquette operators $P_{\hat{a}}$</u>

Call each square made by 4 edges a <u>face</u> of the lattice, and label by $\hat{a}, \hat{b}, \hat{c}, \dots \in \{1, \dots, n^2\}$



$\hat{a}$

& write "$\ell \in \hat{a}$" if edge ℓ is an edge of face $\hat{a}$.

Then

$$\boxed{P_{\hat{a}} \doteq \prod_{\ell \in \hat{a}} X_\ell = X_{\ell_1} X_{\ell_2} X_{\ell_3} X_{\ell_4}}$$

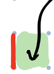
where X_ℓ = Pauli X-gate acting on ℓ^{th}-qubit.

$$\Rightarrow \quad P_{\hat{a}} P_{\hat{b}} = P_{\hat{b}} P_{\hat{a}} \qquad \boxed{\prod_{\hat{a}=1}^{n^2} P_{\hat{a}} = I}$$

$$P_{\hat{a}}^2 = I \qquad \Rightarrow n^2-1 \text{ independent } \{P_{\hat{a}}\}\text{'s.}$$

eigenvalues $\in \{\pm 1\}$

eigenstates ?

$P_{\hat{a}}$ () =  $\hat{a}$

$P_{\hat{a}}$ () = 

$P_{\hat{a}}$ () = 

flips color of edges of face

⋮ etc

$\therefore P_{\hat a}$ eigenstates =

$$P_{\hat a}\ \frac{1}{\sqrt{2}}\left(\quad + \quad \right) = +\frac{1}{\sqrt{2}}\left(\quad + \quad \right)$$

$$P_{\hat a}\ \frac{1}{\sqrt{2}}\left(\quad - \quad \right) = -\frac{1}{\sqrt{2}}\left(\quad - \quad \right)$$

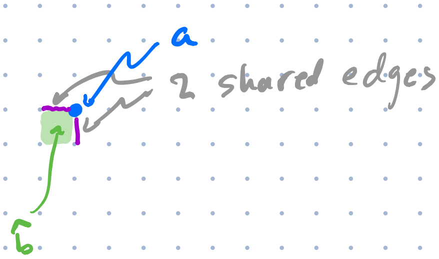
on each face.

- <u>Vertex & Plaquette operators commute</u>

$$[V_a, P_{\hat b}] = 0 \qquad \forall a \quad \forall \hat b$$

– If vertex a & face $\hat b$ have no edge in common, then obvious

– If they share an edge, then must share exactly 2 edges:



In this case

$$[V_a, P_{\hat b}] \propto [Z_{\ell_1} Z_{\ell_2}, X_{\ell_1} X_{\ell_2}] = Z_{\ell_1} Z_{\ell_2} X_{\ell_1} X_{\ell_2} - X_{\ell_1} X_{\ell_2} Z_{\ell_1} Z_{\ell_2}$$

$$= Z_{e_1} X_{e_1} Z_{e_2} X_{e_2} - X_{e_1} Z_{e_1} X_{e_2} Z_{e_2}$$

$$= (-X_{\ell_1} Z_{\ell_1})(-X_{\ell_2} Z_{\ell_2}) - X_{\ell_1} Z_{\ell_1} X_{\ell_2} Z_{\ell_2} = 0. \checkmark$$

- Since all $F_n \in \{ V_a, P_{\hat{a}} \}$ commute, there is an orthonormal basis of simultaneous eigenstates of all F_n.

- There are $2n^2$ qubits $\Rightarrow$ 2^{2n^2}-dim'l state space. There are

$$(n^2-1) + (n^2-1) = 2n^2 - 2$$

independent $\{F_n\} \equiv \{V_a, P_{\hat{a}}\}$ operators,

$\Rightarrow$ 2^{2n^2-2} different possible patterns of ± 1 eigenvalues of $\{F_n\}$

$$\Rightarrow \boxed{\text{simultaneous eigenspaces} = 2^2\text{-dimensional}}$$

- Thus $\{F_n\}$ = error syndrome for 2 qubits encoded in $2n^2$ qubits.

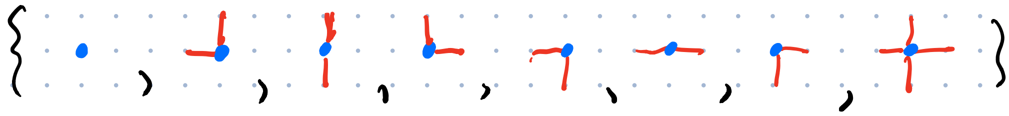
- <u>The code subspace</u>

  $\therefore$ The subspace w/ all $V_a = P_{\hat{a}} = 1$ is 4-dimensional. Call this the

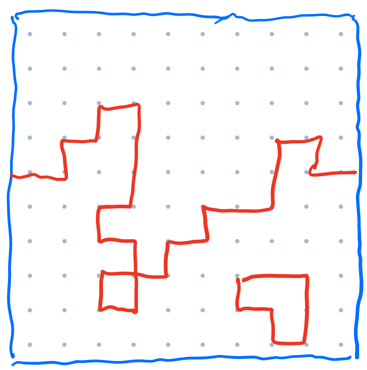
code subspace.

- What is an orthonormal basis?

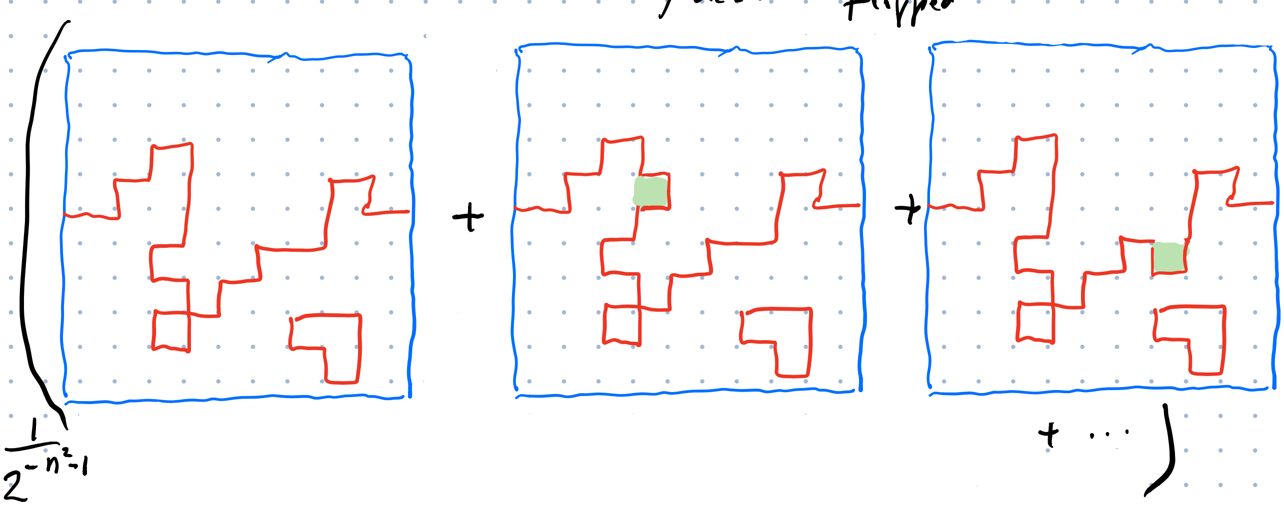
— $V_a = 1 \quad \forall a \Rightarrow$ even # of — edges at each vertex, i.e.

$\in$ 

$\Rightarrow$ the red edges form closed loops

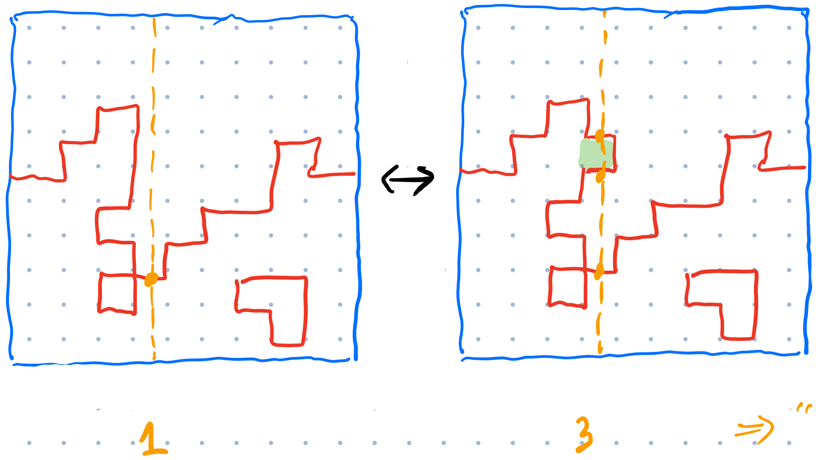
e.g. 

— $P_{\hat{a}} = 1 \quad \forall \hat{a} \Rightarrow$ for each state make superposition w/ all faces 'flipped'

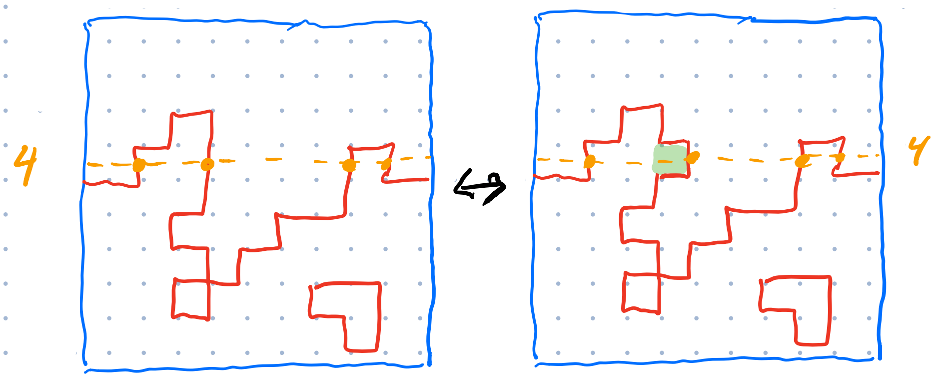


$\frac{1}{2^{-n^2-1}}$

This flipping operation doesn't change the <u>parity</u> (even or odd) of the number of red edges intersecting any line along the x- or the y-axis:



$\Rightarrow$ "$n_x = 1$" (odd)



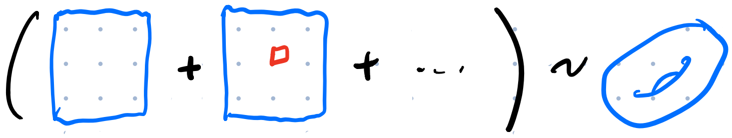
$\Rightarrow$ "$n_y = 0$" (even)

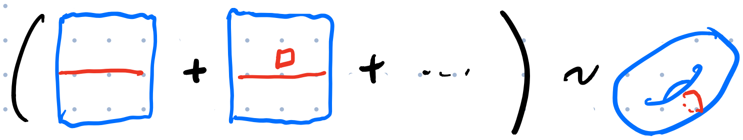
- So 4 "code states"

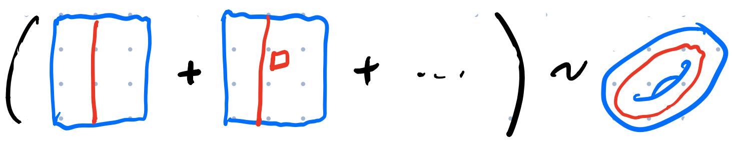
$$|n_x n_y\rangle_L \in \{|00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L\}$$

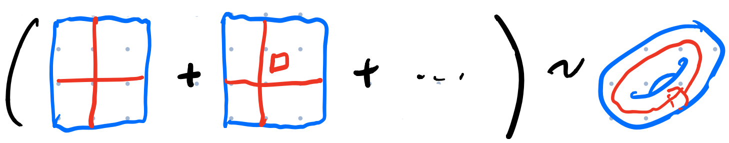
- <u>Exercise</u>: show that they are orthogonal.

So, our 2-qubit code space is spanned by:

$|00\rangle_L \propto$  no wraps

$|10\rangle_L \propto$  1-wrap x-dir

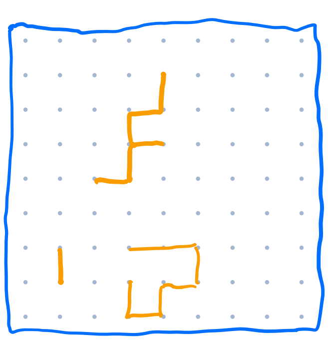
$|01\rangle_L \propto$  1-wrap y-dir

$|11\rangle_L \propto$  1-wrap x+y-dirs

- <u>Error Correction</u>

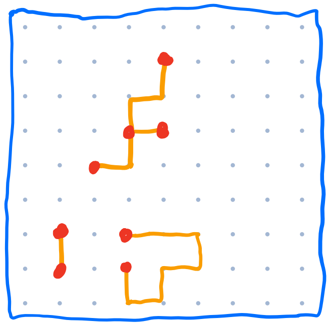
  Look first at X_ℓ & Z_ℓ errors, then at general errors.

- <u>X errors</u>: mark an edge where an X-error occurs with —:



$$= \left(\prod_\ell X_\ell\right) |\psi\rangle_L$$ ← in code subspace

Measure V_a's: get $V_a = -1$ on vertices at "ends" of strings of X-errors.



● = vertices where $V_a = -1$

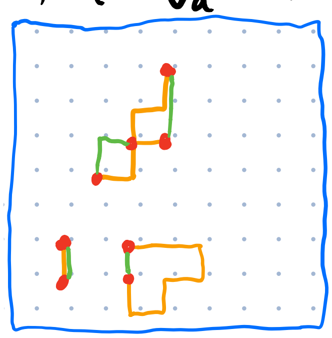
This is because $V_a = \prod_{\ell \in a} Z_\ell$

& $\quad Z_\ell X_\ell = -X_\ell Z_\ell$

So if an odd number of X_ℓ's act for $\ell \in a$, then

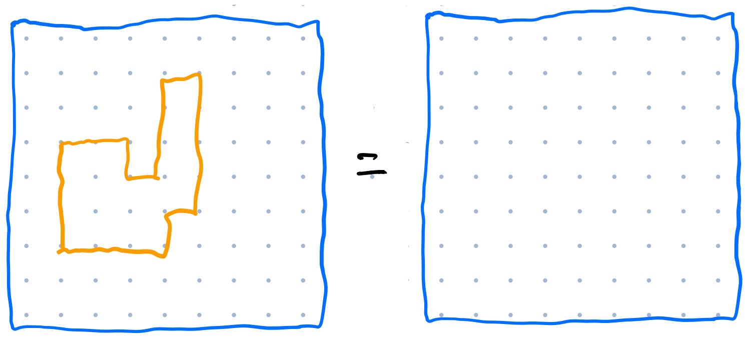
$$V_a \left(\prod X_\ell\right)|\psi\rangle_L = -\left(\prod X_\ell\right) V_a |\psi\rangle_L = -\left(\prod X_\ell\right)|\psi\rangle_L .$$

- So any configuration of X-error strings with end points is detectable by measuring the V_a's.
- Correct these errors by then acting with X_ℓ's on any string of edges connecting the $V_a = -1$ vertices. E.g.:



— = edges where we act w/ X_ℓ's to correct error.

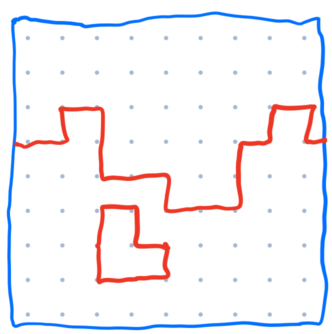
This works since any <u>closed</u> path of X_e's acts as the <u>identity</u> on the code space: e.g.



$= |\psi\rangle_L$

To see this, look at any given computational basis state contributing to $|\psi\rangle_L$, e.g.

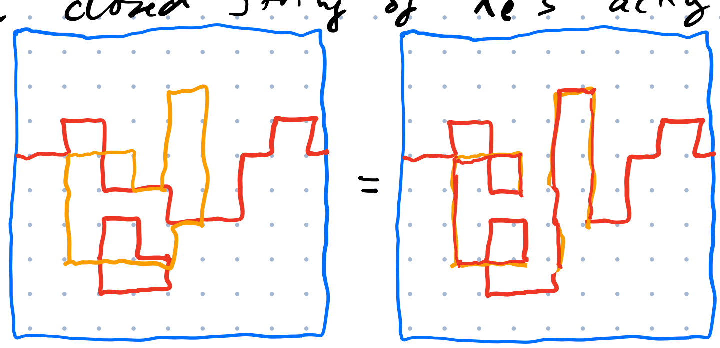
$|10\rangle_L \ni$



(where here — = edge in $|1\rangle$ state)

Then closed string of X_e's acting gives

$\left(\prod_e X_e\right)|10\rangle_L \ni$

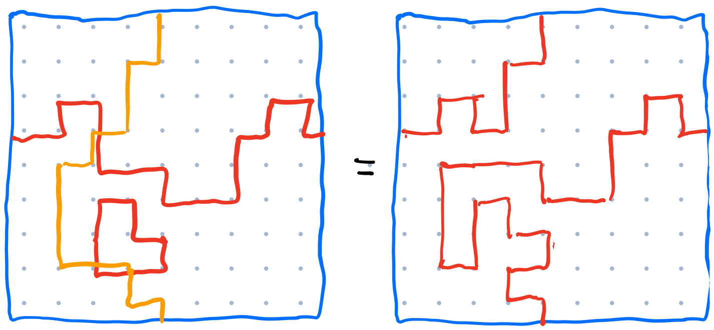


$\in |10\rangle_L$

which is just another term in $|\psi\rangle_L$ since it gives another configuration with a closed string of $|1\rangle$ (or —) edges.

- But this argument breaks down if the closed string of X_e's stretches across the whole $n \times n$ square of qubits (i.e., wraps around a cycle of the torus): e.g.

$(\prod_e X_e)|10\rangle_L \supset$

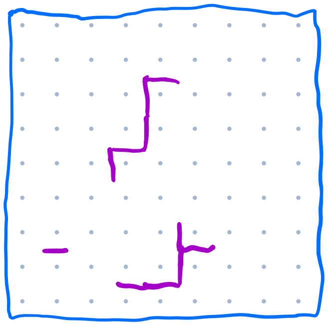


$\subset |11\rangle_L$

changed logical state!

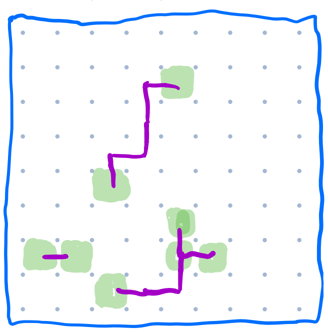
- Thus we cannot correct for X-errors if there are so many of them that they form a string around either (or both) cycles of the torus. But we need at minimum n such errors to wrap the torus, so <u>we can correct in this way up to n X-errors</u>.

• **Z-errors** This is very similar to the X errors. Mark an edge "—" where a Z error occurs by "+", i.e. by drawing a purple interval bisecting it perpendicularly. E.g.



$$= \left(\prod_{\ell} Z_\ell\right) |\psi\rangle_L$$ in code subspace

Measure $P_{\hat{a}}$'s: get $P_{\hat{a}} = -1$ on faces at "ends" of purple strings of Z-errors:

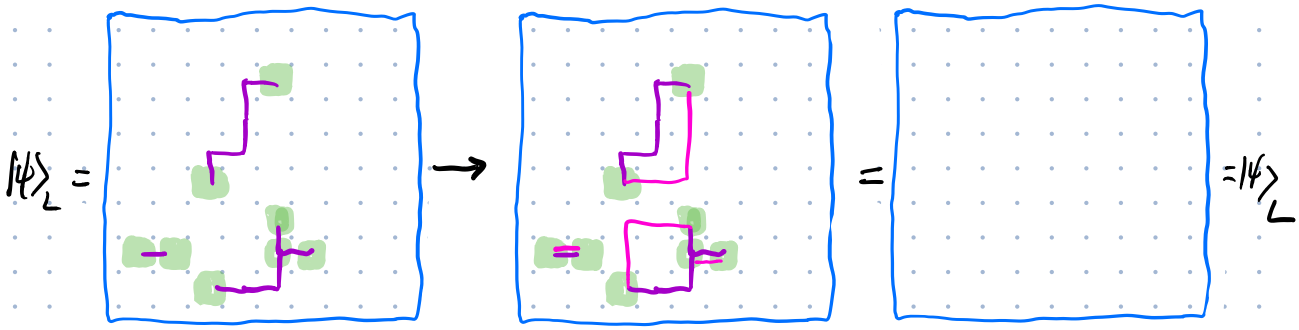


■ = faces when $P_{\hat{a}} = -1$.

This is because if an odd number of Z_ℓ's with $\ell \in \hat{a}$ then

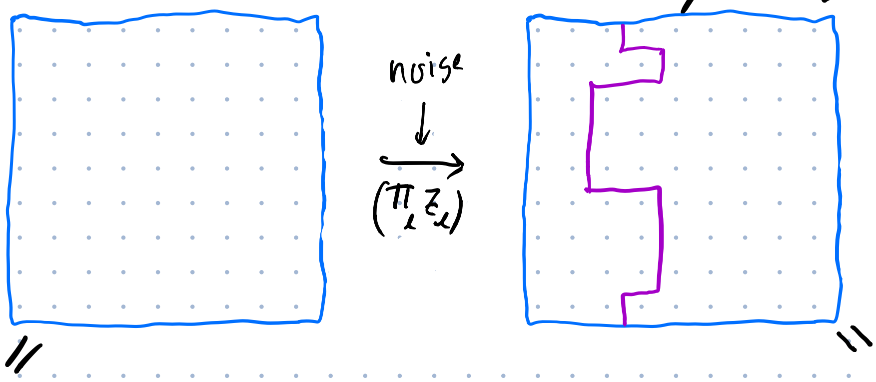
$$P_{\hat{a}} \left(\prod Z_\ell\right) |\psi\rangle_L = -\left(\prod Z_\ell\right) P_{\hat{a}} |\psi\rangle_L = -\left(\prod Z_\ell\right) |\psi\rangle_L.$$

• Correct these errors by acting with <u>any</u> Z_ℓ's to close the purple paths:



where the last equality follows by a similar argument as for closed loops of X-errors.

- Again, this doesn't work if the loop of Z errors stretches around one or both cycles of the torus:



$$a|00\rangle_L + b|10\rangle_L + c|01\rangle_L + d|11\rangle_L \neq a|00\rangle_L - b|10\rangle_L + c|01\rangle_L - d|11\rangle_L$$

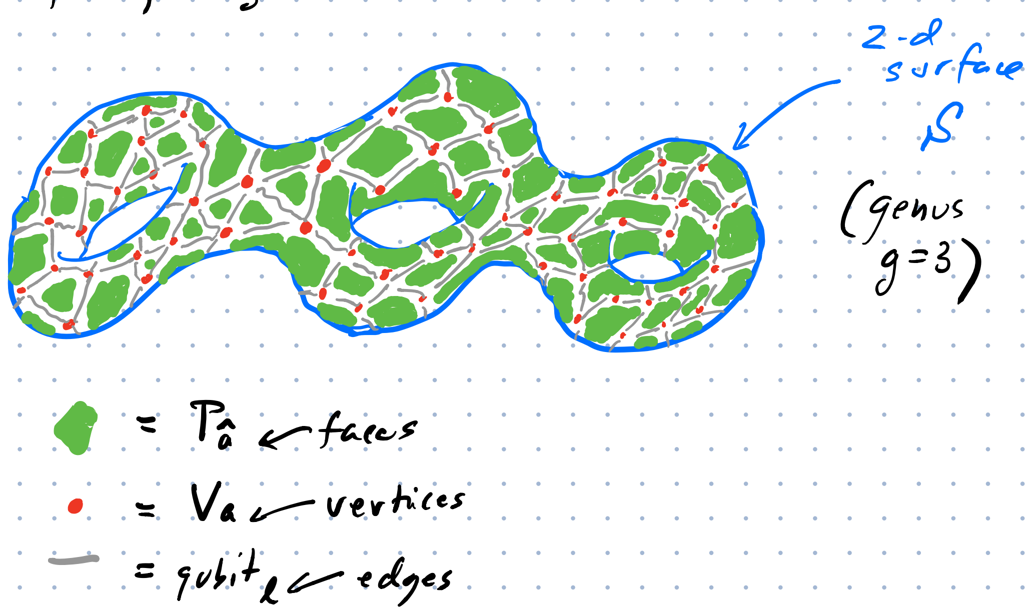
So again, <u>can correct for all up to n Z-errors</u>.

- <u>Y-errors</u>. Exactly as with discussion of Shor's code, since $Y_\ell = i X_\ell Z_\ell$ & we can correct for X_ℓ & Z_ℓ errors independently, we automatically correct for up to n Y_ℓ errors.

- E_k-errors. General errors are give by error operators $E_k = aI + bX + cY + dZ$. By same argument as in Shor's code, if our code corrects for X, Y, & Z errors, it also automatically corrects for E_k errors.

- Generalization

Above did not depend on having a square lattice. It works just as well for any 2-dimensional "graph" of qubits, i.e. for any arrangement of qubits on a surface, e.g.



And code subspace w/ $P_{\hat{a}} = V_a = +1$ $\forall a$ encodes $2g$ qubits where

g = genus of surface S

= # of "handles"

Only depends on topology of S.

TOPOLOGICAL QUANTUM COMPUTING

Idea #1: realize the large set of encoding qubits as the states of a macroscopic material.

E.g. all the $2n^2 = N$ qubits of the toric code might be electron spin states on the surface of a metal in the shape of a donut.

Even if it is macroscopic, so

$$N \sim 10^{20}$$

it would still only encode 2 code qubits.

Idea #2: Arrange the interaction energy

among the N encoding qubits so that noise is automatically suppressed — i.e. you pay a large energy cost to add 1-qubit noise to the system.

E.g. Here is a Hamiltonian (energy function) for the toric code encoding qubits which does the job:

$$H = -E_0 \Big(\sum_{a \in \text{vertices}} V_a + \sum_{\hat{a} \in \text{faces}} P_{\hat{a}} \Big)$$

- Since the eigenvalues of V_a, $P_{\hat{a}}$ are all ± 1 and they all commute, the possible eigen-energies of H are

$$E = -E_0 \Big(\sum_a (\pm 1) + \sum_{\hat{a}} (\pm 1) \Big).$$

- Clearly the lowest-energy states are those with $V_a = P_{\hat{a}} = +1 \quad \forall a, \hat{a}$

w/ energy $E = -2n^2 E_0$.

Note that these states are the toric code subspace, so they are precisely the 4-dim'l space with basis

$$\{ |00\rangle_L, |01\rangle_L, |10\rangle_L, |11\rangle_L \}.$$

- Flipping a qubit to make some V_a or $P_{\hat{a}} = -1$ costs an energy E_0, so if E_0 can be made large enough, noise will be automatically suppressed.

[Physically, the probability of a transition costing an energy E_0 is typically suppressed by an exponential factor $e^{-\beta E_0}$, where the energy scale

$$k_B T \equiv \frac{1}{\beta}$$

is called the "noise temperature". It is often the actual temperature of the system. So for given E_0, noise in such a system can be exponentially suppressed by making the temperature small enough.]

- Note that the Hamiltonian is <u>local</u> on the lattice of encoding qubits, because $P_{\hat{a}}$ and V_a are products of operators acting on "nearest neighbor" qubits. This is typical of actual materials, and gives some hope that such a system can be realized.

- <u>Insensitivity to the detailed interactions</u>

<u>Claim</u>: If you perturb the Hamiltonian by <u>any</u> local operator

$$H \to H + \delta H$$

$$\left(\delta H = \sum_a \mathcal{O}_a \, \epsilon \quad \text{with} \quad |\epsilon| \ll E_0 \right)$$

then:

- Ground states $\{|n_x n_y\rangle_L\}$ remain degenerate (to exponential accuracy)

- Still have finite energy gap $\sim E_0$ protecting against local noise.

[Reason: From analysis of toric code, know we need accumulation of $\sim \sqrt{N}$ local operator actions ("errors") to change the code state. Thus δH needs to act $\sim \sqrt{N}$ times to modify ground state energy implying it lifts the degeneracy of the code subspace by an amount

$$\Delta E \sim \frac{1}{(\sqrt{N})!}\left(\frac{\epsilon}{E_0}\right)^{\sqrt{N}}$$

(this estimate comes from perturbation theory.)

Note $\sqrt{N} \propto$ linear size of system,
&

$$\sqrt{N} \sim L/a \leftarrow \text{atomic spacing},$$

so for macroscopic systems, the code subspace remains effectively degenerate.

- This means we don't need to exquisitely engineer our topological computer material: it just has to be "close enough". This gives hope that such materials might actually exist in nature.

→ In fact, they do! Since around 1990 it was realized that certain states of matter have this kind of property. They are known as "topological phases of matter".

The best-known example are simply the surface electron states of a metal in a magnetic field at low temperatures. (An indication of the topological order is the observed "quantum Hall effect" behavior of these materials.) By now many different examples of topological phases are known.

- So, there are materials that "automatically" protect code qubits from noise. But how do we do computations with them? I.e., what are the analog of quantum gates and circuits for such materials?

Idea #3: Use "noise" as a computational resource!

- In these topological phases, the "gap" — the high energy price, E_0, that one pays to excite the system above its ground states — can be used as a tool to control the specific state of the code subspace (the 4-dimensional subspace of ground states in the toric code case).

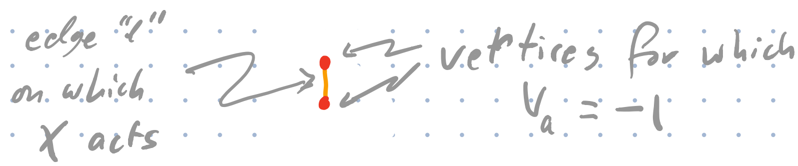
- Consider an energy eigenstate in which $V_{a_0} = -1$ for one vertex a_0, and otherwise $V_a = P_{\hat{a}} = +1$ for all the other vertices and all the faces. This is a state localized at vertex a_0 with energy $2E_0$ above the ground state. We can think of this as a "V-particle" at a_0 (Physicists often call such states of localized energy "quasi-particles".)

- According to our topological Hamiltonian

$$H = -E_0 \left(\sum_a V_a + \sum_{\hat{a}} P_{\hat{a}} \right)$$

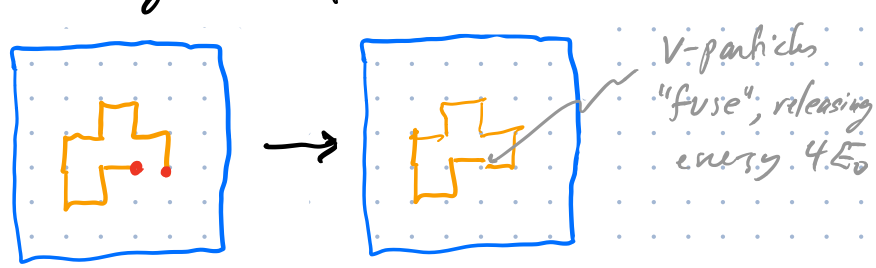
the energy of the V-particle does not depend on its location a_0. So, once we have expended energy $2E_0$ to create it, it costs almost no energy to move it around.

- Consider now adding localized energy to flip a single qubit $|\psi\rangle_L \to X_\ell |\psi\rangle_L$

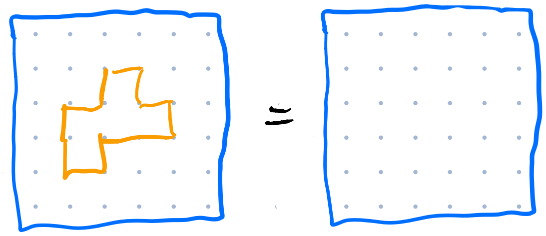


This has the effect of creating two neighboring V-particles as shown above (recall the discussion of X-noise earlier), thus costing energy $4E_0$.

- We can now move the two V-particles at will along some path



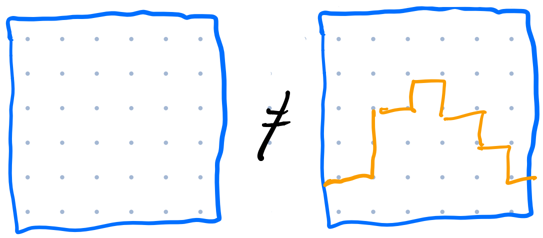
at no cost in energy. If we bring them back together we can cancel them (gaining back energy $4E_0$) and leaving the original state $|\psi\rangle_L$ changed by a closed path of spin-flips. But as we showed earlier, this does not change $|\psi\rangle_L$:



$= |\psi\rangle_L$

as long as the closed path of spin-flips does not wrap one or both cycles of the torus:

$|n_x, n_y\rangle_L =$



$= |n_x \oplus 1, n_y\rangle_L$

- Thus pair-creating V-particles, moving one of them around a cycle & then annihilating them acts as a "gate" transforming the logical qubits.

- We can also make a $|\psi\rangle_L \to Z_L|\psi\rangle_L$ phase flip. This creates two neighboring "P-particles" — localized states where $P_{\hat{a}_0} = -1$ for a face $\hat{a}_0$. Again, moving one P-particle around a cycle of the torus & annihilating it gives a phase transformation of $|\psi\rangle_L$.

- Similarly, a $|\psi\rangle_L \to Y_e|\psi_L\rangle \propto X_e Z_e|\psi_L\rangle$ creates a pair of "PV-particles" — a bound state of a P- & a V-particle.

- For topological matter realizing the toric code, these are all the independent excitations at our disposal, so we can only make gates by pair-creating P, V, or PV particles, moving them around, then annihilating them.

- It turns out that the set of 2-qubit gates one obtains in this way is <u>not</u> rich enough to permit general quantum computations. Essentially, the problem is that moving any of these particles along a path looping around another particle ("braiding" the particle paths) all we can get are simple phases

$$E_j E_k \xrightarrow{\text{braid}} e^{i\theta_{jk}} E_k E_j$$

where $E_j \in \{P, V, \text{or } PV\}$ are the different possible excitations.

In the case of the toric code,

$$e^{i\theta_{jk}} = \begin{array}{c|ccc} & P & V & PV \\ \hline P & +1 & +1 & +1 \\ V & +1 & +1 & +1 \\ PV & +1 & +1 & -1 \end{array}$$

meaning that P- & V-particles act like bosons and PV-particles act like fermions.

- To be able to use the excitations of a topological phase to construct general quantum gates, it turns out that one needs the excitations to satisfy more general "non-abelian braid statistics"

$$E_j E_k \xrightarrow{\text{braid}} B^{(lm)}_{(jk)} E_l E_m$$

for some complex matrix $B^{(lm)}_{(jk)}$ with non-zero entries when $(lm) \neq (kj)$.

Metals at very low temperatures, high magnetic fields, and subject to an external electric potential such that they are at certain "fraction quantum Hall plateaus" do show such non-abelian braid statistics.!