# Axioms of Quantum Mechanics

Underlined terms are linear algebra concepts whose definitions you need to know.
*Italicized terms* are the concepts being defined by the axioms.

**I.** The *state* of a system is a vector, $|\psi\rangle$, in a Hilbert space, $\mathcal{H}$ (a complex vector space with a positive definite inner product), and is normalized: $\langle\psi|\psi\rangle = 1$. Also, the phase of the state is unobservable, so if $|\psi\rangle = e^{i\alpha}|\chi\rangle$, then $|\psi\rangle$ and $|\chi\rangle$ describe the same physical state of the system. (This can be summarized by saying that a state is a ray in a Hilbert space.)

**II.** An *observable* (allowed measurement) is a choice of a hermitian operator, $\widehat{M}$. By the spectral theorem, $\widehat{M} = \sum_i \mu_i \widehat{P}_i$, where $\mu_i$ are its eigenvalues and $\widehat{P}_i$ are the orthogonal projection operators onto their corresponding eigenspaces. Examples of observables are energy, position, momentum, and angular momentum operators, which are basically all the observables we will use in the course.

**III.** The only possible *outcomes* of measuring $\widehat{M}$ are one of its eigenvalues. I denote this outcome of this measurement by "$M = \mu_i$".

**IV.** The *probability of observing a given possible outcome*, $M = \mu_i$, of such a measurement is denoted $\mathcal{P}(M{=}\mu_i)$, and is given by the squared norm of the projection of the state onto the eigenspace of the eigenvalue $\mu_i$. In formulas, this is

$$\mathcal{P}(M{=}\mu_i) = \left\| \widehat{P}_i|\psi\rangle \right\|^2 = \langle\psi|\widehat{P}_i|\psi\rangle. \tag{1}$$

This is known as the *Born rule.*

**V.** Once we *observe* or *measure* the outcome $M = \mu_i$, the state changes as a result of the measurement from its state $|\psi\rangle$ immediately before the measurement, to a new state $|\psi'\rangle$ immediately after the measurement, given by its normalized projection onto the eigenspace corresponding to the observed eigenvalue. In formulas, this is

$$|\psi\rangle \xrightarrow{\text{meas. } M=\mu_i} |\psi'\rangle = \frac{\widehat{P}_i|\psi\rangle}{\sqrt{\mathcal{P}(M{=}\mu_i)}} = \frac{\widehat{P}_i|\psi\rangle}{\left\|\widehat{P}_i|\psi\rangle\right\|}. \tag{2}$$

This rule really only applies to idealized instantaneous nondestructive measurements, also known as *projective measurements.* The results of real-life measurements are typically much more complicated to describe, but their effect on the observed state is always in some sense greater than that of the ideal projective measurement shown in (2).

**VI.** The *time evolution* of the state of an isolated system (i.e., when it is not being measured or otherwise interacting with the external world) is given by $|\psi(t)\rangle = \widehat{U}(t)|\psi(0)\rangle$, where the unitary *time evolution operator* is given by $\widehat{U}(t) = \exp\{-it\widehat{H}/\hbar\}$ where $\widehat{H}$ is the hermitian energy operator (also known as the Hamiltonian operator).

**VII.** If a system can be *decomposed* into two subsystems each respectively described by states in Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, then states of the combined system are in the tensor product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$.