Problem Set 6

Each problem is worth the points indicated.

Problem 1. (3 points) Find the potential in an empty 2-dimensional region (i.e., ignore the 3rd, z, coordinate) described by $s_1 \le s \le s_2$ and $0 \le \phi \le \pi$ where (s, ϕ) are polar coordinates in the x-y plane and s_0 and s_1 are positive constants. The potential is given on the boundaries as $V = 0$ at $s = s_1$ and $s = s_2$, $V = V_0$ at $\phi = 0$, and $V = V_\pi$ at $\phi = \pi$, where V_0 and V_π are given constants.

Solution: In (s, ϕ) polar coordinates in the plane, Laplace's equation is

$$0 = \nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2}.$$

With the separation of variables ansatz $V = A(s)B(\phi)$ this becomes, after multiplying by s^2 and dividing by AB,

$$0 = \frac{s}{A}\frac{d}{ds}\left(s\frac{dA}{ds}\right) + \frac{1}{B}\frac{d^2B}{d\phi^2} = \frac{1}{A}\frac{d^2A}{(d\ln s)^2} + \frac{1}{B}\frac{d^2B}{d\phi^2},$$

where in the last step I used that $s\,d/ds = d/d\ln s$. Since this is a sum of terms which depend on different variables, it can only be satisfied if the terms are individually constant, so

$$\frac{1}{A}\frac{d^2A}{(d\ln s)^2} = -k^2, \qquad\qquad \frac{1}{B}\frac{d^2B}{d\phi^2} = +k^2,$$

for some real constant k^2. (This means that k can be real or imaginary.) The general solutions of these differential equations are

$$A = a\,\sin(k\ln s) + b\,\cos(k\ln s), \qquad\qquad B = c\,\sinh(k\phi) + d\,\cosh(k\phi),$$

where a, b, c, d are undetermined constants. So the general solution for V of the Laplace equation can be written as the sum

$$V(s,\phi) = \sum_k \left[a_k\,\sin(k\ln s) + b_k\,\cos(k\ln s)\right]\cdot\left[c_k\,\sinh(k\phi) + d_k\,\cosh(k\phi)\right], \tag{1}$$

where the sum is over some as-yet-undetermined set of real or imaginary values of k.

Impose the $s = s_i$ boundary conditions first. At $s = s_1$ we have from (1)

$$0 = V(s_1,\phi) = \sum_k \left[a_k\,\sin(k\ln s_1) + b_k\,\cos(k\ln s_1)\right]\cdot\left[c_k\,\sinh(k\phi) + d_k\,\cosh(k\phi)\right].$$

Note that an overall factor in the first factor can be absorbed in the second factor by a redefinition of the a, b, c, d constants. Since the $\sinh(k\phi)$ and $\cosh(k\phi)$ functions are independent for all k, the only way this sum can vanish for all ϕ is if the coefficients vanish. Thus $a_k\,\sin(k\ln s_1) + b_k\,\cos(k\ln s_1) = 0$ for all k, implying $a_k/b_k = -\cos(k\ln s_1)/\sin(k\ln s_1)$. So, after redefining the c_k and d_k coefficients suitably, the general solution for V can be written

$$\begin{aligned} V(s,\phi) &= \sum_k \left[\sin(k\ln s)\,\cos(k\ln s_1) - \cos(k\ln s)\,\sin(k\ln s_1)\right]\cdot\left[c_k\,\sinh(k\phi) + d_k\,\cosh(k\phi)\right], \\ &= \sum_k \sin(k(\ln s - \ln s_1))\left[c_k\,\sinh(k\phi) + d_k\,\cosh(k\phi)\right] = \sum_k \sin(k\ln(s/s_1))\left[c_k\,\sinh(k\phi) + d_k\,\cosh(k\phi)\right], \end{aligned}$$

where in the last two steps I used a trig identity and then a log identity to simplify the expression. Now impose the $s = s_2$ boundary condition to get

$$0 = V(s_2, \phi) = \sum_k \sin(k \ln(s_2/s_1)) \left[c_k \sinh(k\phi) + d_k \cosh(k\phi)\right].$$

Since the $\sinh(k\phi)$ and $\cosh(k\phi)$ functions are independent for all k, the only way this sum can vanish for all ϕ is if the coefficients vanish. Thus $\sin(k \ln(s_2/s_1)) = 0$, implying $k = n\pi/\ln(s_2/s_1)$, for $n \in \mathbb{Z}$, and the general solution for V can be written

$$V(s, \phi) = \sum_{n=1}^{\infty} \sin(n\pi \ln(s/s_1)/L) \left[c_n \sinh(n\pi\phi/L) + d_n \cosh(n\pi\phi/L)\right], \quad \text{where} \quad L \doteq \ln(s_2/s_1). \tag{2}$$

I only included the sum over positive n since the negative n terms are not independent.

Now impose the $\phi = 0, \pi$ boundary conditions. At $\phi = 0$ we have

$$V_0 = V(s, 0) = \sum_{n=1}^{\infty} \sin(n\pi \ln(s/s_1)/L) d_n. \tag{3}$$

We solve for the d_n by integrating both sides of (3) against $(2/L) \int_0^L d\ln(s/s_1) \sin(m\pi \ln(s/s_1)/L)$ and using the orthogonality of the sine functions

$$(2/L) \int_0^L d\ln(s/s_1) \sin(m\pi \ln(s/s_1)/L) \sin(n\pi \ln(s/s_1)/L) = \delta_{n,m}.$$

This gives

$$d_n = \frac{2}{L} \int_0^L d\ln(s/s_1)\, \sin(n\pi \ln(s/s_1)/L)\, V_0 = \frac{2V_0}{n\pi} \left[1 - (-)^n\right]. \tag{4}$$

Now do the same with the $\phi = \pi$ boundary condition to get

$$V_\pi = V(s, \pi) = \sum_{n=1}^{\infty} \sin(n\pi \ln(s/s_1)/L) \left[c_n \sinh(n\pi^2/L) + d_n \cosh(n\pi^2/L)\right].$$

Using the same sine function orthogonality gives

$$c_n \sinh(n\pi^2/L) + d_n \cosh(n\pi^2/L) = \frac{2V_\pi}{n\pi} \left[1 - (-)^n\right]. \tag{5}$$

(4) and (5) imply

$$c_n = \frac{2\left[1 - (-)^n\right]}{n\pi \sinh(n\pi^2/L)} \left(V_\pi - V_0 \cosh(n\pi^2/L)\right).$$

Plugging this and (4) into (2) gives, after a little algebra, the final answer

$$\begin{aligned}
V(s, \phi) &= \sum_{n=1 \ \text{odd}}^{\infty} \sin\left(\frac{n\pi}{L} \ln \frac{s}{s_1}\right) \left[\frac{4V_\pi - 4V_0 \cosh(n\pi^2/L)}{n\pi \sinh(n\pi^2/L)} \sinh\left(\frac{n\pi}{L}\phi\right) + \frac{4V_0}{n\pi} \cosh\left(\frac{n\pi}{L}\phi\right)\right], \\
&= \sum_{n=1 \ \text{odd}}^{\infty} \frac{4}{n\pi} \frac{1}{\sinh(n\pi^2/L)} \sin\left(\frac{n\pi}{L} \ln \frac{s}{s_1}\right) \left[\left[V_\pi - V_0 \cosh(n\pi^2/L)\right] \sinh\left(\frac{n\pi}{L}\phi\right) + \left[V_0 \sinh(n\pi^2/L)\right] \cosh\left(\frac{n\pi}{L}\phi\right)\right], \\
&= \sum_{n=1 \ \text{odd}}^{\infty} \frac{4}{n\pi} \frac{1}{\sinh(n\pi^2/L)} \sin\left(\frac{n\pi}{L} \ln \frac{s}{s_1}\right) \left[V_\pi \sinh\left(\frac{n\pi}{L}\phi\right) + V_0 \sinh\left(\frac{n\pi}{L}(\pi - \phi)\right)\right],
\end{aligned}$$

where in the first step I used that $1-(-)^n$ is 0 for n even and 2 for n odd, in the last step I used a (hyperbolic) trig identity, and recall that L is my shorthand for $\ln(s_2/s_1)$.

Problem 2. (1 point) Find the potential in a rectangular parallelepiped (a 3d rectangular box, also known as a rectangular cuboid) of sides of lengths a, b, and c. The potential is set to $V = V_1$ on one of the faces with sides b and c, and is zero on the remaining five faces.

Solution: This is very similar to the cube problem done in the notes. As in that problem, set up the problem with an a, b, and c edge along the positive x, y, and z axis, respectively, and so the $V = V_1$ face is at $x = a$. Then separation of variables will give (hyperbolic) sines and cosines in each variable, and imposing the $y = 0, b$ and $z = 0, c$ boundary conditions where $V = 0$ will impose that only sines remain for those terms, giving

$$V(x, y, z) = \sum_{n,m\in\mathbb{Z}} \sin(n\pi y/b)\sin(m\pi z/c)\left[A_{n,m}\sinh(k_{n,m}x) + B_{n,m}\cosh(k_{n,m}x)\right],$$

where $k_{n,m}^2 = (n\pi/b)^2 + (m\pi/c)^2$, or,

$$k_{n,m} = \pi\sqrt{\frac{n^2}{b^2} + \frac{m^2}{c^2}}. \tag{6}$$

Imposing the $x = 0$ boundary condition where $V = 0$, implies $B_{n,m} = 0$, so

$$V(x, y, z) = \sum_{n,m\in\mathbb{Z}} \sin(n\pi y/b)\sin(m\pi z/c)A_{n,m}\sinh(k_{n,m}x). \tag{7}$$

Finally, the $x = a$ boundary condition gives

$$V_1 = \sum_{n,m\in\mathbb{Z}} \sin(n\pi y/b)\sin(m\pi z/c)A_{n,m}\sinh(k_{n,m}a).$$

The coefficients are determined by using the orthogonality of the sine functions to give

$$A_{n,m}\sinh(k_{n,m}a) = \frac{4V_1}{bc}\frac{bc}{nm\pi^2}[1-(-)^n][1-(-)^m].$$

Plugging that into (7) gives the final answer

$$V(x, y, z) = \frac{16V_1}{\pi^2}\sum_{n,m\ \text{odd}} \frac{1}{nm}\frac{\sinh(k_{n,m}x)}{\sinh(k_{n,m}a)}\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{c}x\right),$$

where $k_{n,m}$ is defined in (6).

Problem 3. (3 points) Find the potential in an empty region described by $a \le r \le b$ in 3-dimensional spherical coordinates, where a and b are positive constants. The potential is held at zero on both the $r = a$ and $r = b$ positive-z hemispheres, is held at $V = V_a$ on the $r = a$ negative-z hemisphere, and is held at $V = V_b$ on the $r = b$ negative-z hemisphere. You do not need to give closed-form algebraic expressions for the coefficients of the separation-of-variables series solution, but you do need to give suitable integral expressions for them.

Solution: Use separation of variables in spherical coordinates. Since the problem has rotational symmetry around the z-axis and the boundaries are complete spheres, we can use our general result

$$V(r, \theta) = \sum_{\ell=0}^{\infty}\left[A_\ell r^\ell + B_\ell r^{-\ell-1}\right]P_\ell(\cos\theta). \tag{8}$$

The boundary conditions are $V(a,\theta) = V_a H(\theta)$ and $V(b,\theta) = V_b H(\theta)$ where I have defined the function

$$H(\theta) \doteq \begin{cases} 0 & 0 \le \theta \le \frac{\pi}{2} \\ 1 & \frac{\pi}{2} < \theta \le \pi \end{cases} .$$

Imposing the $r = a$ boundary condition gives

$$V_a H(\theta) = \sum_\ell [A_\ell a^\ell + B_\ell a^{-\ell-1}] \, P_\ell(\cos\theta).$$

Integrating both sides of this equation against $\frac{1}{2}(2m+1)\int_0^\pi \sin\theta d\theta \, P_m(\cos\theta)$ and using orthogonality of the Legendre polynomials --- $\frac{1}{2}(2m+1)\int_0^\pi \sin\theta d\theta \, P_m(\cos\theta)\, P_\ell(\cos\theta) = \delta_{m,\ell}$ --- gives

$$A_\ell a^\ell + B_\ell a^{-\ell-1} = V_a H_\ell, \tag{9}$$

where I've defined the integrals

$$H_\ell \doteq \frac{2\ell+1}{2}\int_0^\pi \sin\theta d\theta \, H(\theta)\, P_\ell(\cos\theta) = \frac{2\ell+1}{2}\int_{\pi/2}^\pi \sin\theta d\theta \, P_\ell(\cos\theta) = \frac{2\ell+1}{2}\int_{-1}^0 du \, P_\ell(u), \tag{10}$$

where in the last step I changed integration variables to $u = \cos\theta$. The same steps for the $r = b$ boundary gives

$$A_\ell b^\ell + B_\ell b^{-\ell-1} = V_b H_\ell. \tag{11}$$

Solving (9) and (11) for A_ℓ and B_ℓ gives

$$A_\ell = \frac{b^{\ell+1} V_b - a^{\ell+1} V_a}{b^{2\ell+1} - a^{2\ell+1}} H_\ell, \qquad B_\ell = (ab)^{\ell+1} \frac{b^\ell V_a - a^\ell V_b}{b^{2\ell+1} - a^{2\ell+1}} H_\ell. \tag{12}$$

So the general solution is (8) with coefficients given by (12) where the H_ℓ are the number given by the integrals (10).

Problem 4. (1 point) The same as problem 3, but now with a point charge, q added at the origin.

Solution: The solution is exactly the same as in problem 3. Since the point charge is not in the region between the two spheres and since the boundary conditions on that region are the same, the solution is the same.

Problem 5. (2 points) Prove the equivalence of the two forms of the electric field of a pure dipole given in the lecture, $\vec{E}_{\text{dip}}(r,\theta) = p\left[2\cos\theta\,\widehat{r} + \sin\theta\,\widehat{\theta}\right]/(4\pi\epsilon_0 r^3) = [3(\vec{p}\cdot\widehat{r})\widehat{r} - \vec{p}]/(4\pi\epsilon_0 r^3)$.

Solution: The first expression is in spherical coordinates with $\vec{p}$ pointing along the positive z-axis. So $\vec{p} = p\widehat{z}$. So, we want to rewrite $\widehat{z}$ in terms of the spherical unit vectors $\widehat{r}$, $\widehat{\theta}$, and $\widehat{\phi}$. This is something you can just look up, or derive it by inverting eqn. (1.64) of Griffiths as

$$\cos\theta\,\widehat{r} - \sin\theta\,\widehat{\theta} = \cos\theta\,(\sin\theta\cos\phi\,\widehat{x} + \sin\theta\sin\phi\,\widehat{y} + \cos\theta\,\widehat{z}) - \sin\theta\,(\cos\theta\cos\phi\,\widehat{x} + \cos\theta\sin\phi\,\widehat{y} - \sin\theta\,\widehat{z}) = \widehat{z}.$$

Plug this into the numerator of the second expression for $\vec{E}_{\text{dip}}$ to find

$$\begin{aligned} 3(\vec{p}\cdot\widehat{r})\widehat{r} - \vec{p} &= 3p(\widehat{z}\cdot\widehat{r})\widehat{r} - p\widehat{z} = 3p\left[\left(\cos\theta\,\widehat{r} - \sin\theta\,\widehat{\theta}\right)\cdot\widehat{r}\right]\widehat{r} - p\left(\cos\theta\,\widehat{r} - \sin\theta\,\widehat{\theta}\right) \\ &= 3p\cos\theta\,\widehat{r} - p\cos\theta\,\widehat{r} + p\sin\theta\,\widehat{\theta} = p(2\cos\theta\,\widehat{r} + \sin\theta\,\widehat{\theta}), \end{aligned}$$

where in the 3rd step I use orthonormality of the spherical unit vectors.

Problem 6. (1 point) Show that the quadrupole moment tensor $(M_2)_{ij} \doteq Q_{ij}$ is traceless. (Traceless means that considered as a matrix, its trace is zero.)

Solution: The quadrupole moment, $M_2(\widehat{r})$, and the quadrupole moment tensor Q_{ij}, were defined in the lecture to be

$$M_2(\widehat{r}) \doteq \int d\tau' \, (r')^2 \rho(\vec{r}\,') \, P_2(\cos\theta'), \qquad M_2(\widehat{r}) \doteq \sum_{i,j=1}^{3} \widehat{r}_i \widehat{r}_j Q_{ij}, \tag{13}$$

where the second equation defines Q_{ij} implicitly in terms of $M_2(\widehat{r})$. Recall that θ' in the integral expression for $M_2(\widehat{r})$ is defined to be the angle between $\vec{r}\,'$ and $\vec{r}$. This means that $\cos\theta' = \widehat{r}\cdot\widehat{r}'$. Plugging that and the expression for P_2 into the integral expression gives

$$\begin{aligned}
M_2(\widehat{r}) &= \int d\tau' \, (r')^2 \rho(\vec{r}\,') \frac{1}{2} \left(3(\widehat{r}\cdot\widehat{r}')^2 - 1\right) = \frac{1}{2}\int d\tau' \, (r')^2 \rho(\vec{r}\,') \left(3(\widehat{r}\cdot\widehat{r}')^2 - (\widehat{r}\cdot\widehat{r})\right) \\
&= \frac{1}{2}\int d\tau' \, (r')^2 \rho(\vec{r}\,') \left(3(\sum_i \widehat{r}_i \widehat{r}'_i)^2 - \sum_i \widehat{r}_i \widehat{r}_i\right) = \frac{1}{2}\int d\tau' \, (r')^2 \rho(\vec{r}\,') \left(3\sum_{i,j} \widehat{r}_i \widehat{r}'_i \widehat{r}_j \widehat{r}'_j - \sum_{i,j} \widehat{r}_i \delta_{i,j} \widehat{r}_j\right) \\
&= \sum_{i,j} \widehat{r}_i \widehat{r}_j \left[\frac{1}{2}\int d\tau' \, (r')^2 \rho(\vec{r}\,') \left(3\widehat{r}'_i \widehat{r}'_j - \delta_{i,j}\right)\right],
\end{aligned}$$

where in the second step I used $1 = \widehat{r}\cdot\widehat{r}$ to make the expression quadratic in $\widehat{r}$ so it has the form of the second equation in (13), in the third step I wrote out the dot products in an arbitrary Cartesian coordinate system, in the fourth step I rewrote the terms so that each was proportional to $\widehat{r}_i \widehat{r}_j$ which I pulled out of the integral in the last step. Comparing to the second equation in (13) we thus find that the quadrupole moment tensor components are given by the integrals

$$Q_{ij} = \frac{1}{2}\int d\tau' \, (r')^2 \rho(\vec{r}\,') \left(3\widehat{r}'_i \widehat{r}'_j - \delta_{i,j}\right).$$

We now want to show that this matrix is traceless:

$$\begin{aligned}
\mathrm{Tr}Q &= \sum_{i=1}^{3} Q_{ii} = \sum_{i=1}^{3} \frac{1}{2}\int d\tau' \, (r')^2 \rho(\vec{r}\,') \, (3\widehat{r}'_i \widehat{r}'_i - \delta_{i,i}) = \frac{1}{2}\int d\tau' \, (r')^2 \rho(\vec{r}\,') \left(3\left[\sum_{i=1}^{3} \widehat{r}'_i \widehat{r}'_i\right] - \left[\sum_{i=1}^{3} 1\right]\right) \\
&= \frac{1}{2}\int d\tau' \, (r')^2 \rho(\vec{r}\,') \, (3\widehat{r}'\cdot\widehat{r}' - 3) = \frac{1}{2}\int d\tau' \, (r')^2 \rho(\vec{r}\,') \, (3-3) = 0.
\end{aligned}$$

The first equality is the definition of the trace, the 3rd uses that $\delta_{i,i} = 1$, the 4th uses the definition of the dot product, the 5th uses $1 = \widehat{r}'\cdot\widehat{r}'$.

Problem 7. (1 point) What is the dipole moment of six point charges of charge q placed at the centers of the faces of a cube of side L? [*Hint:* trick question!]

Solution: There is no answer: since no origin has been specified and since the monopole moment is non-zero, the dipole moment can have any value you like depending on where you choose the origin of your coordinate system.

Problem 8. (3 points) Find all multipole moments of a sphere of radius R with surface charge density $\sigma(\theta) = \sigma_0 \cos^3\theta$, in spherical coordinates. Sum the resulting multipole expansion series to obtain the potential everywhere outside the sphere in closed form.

Solution: This is very similar to the multipole problem with spherical charge distribution worked in the notes, but with $\sigma_0 \cos\theta$ replaced by $\sigma_0 \cos^3\theta$. So using those results with this substitution we have

$$M_\ell = \sigma_0 R^{\ell+2} \int_0^\pi \sin\theta'\, d\theta' \int_0^{2\pi} d\phi' \, \cos^3\theta \, P_\ell(\cos\theta'),$$
$$\cos\theta = \cos\alpha \, \cos\theta' - \sin\alpha \, \sin\theta' \, \cos\phi',$$

where α is the polar angle of the field vector $\vec{r}$. Now $\int_0^{2\pi} d\phi' \cos^n \phi' = 0$ for n odd, so we only need to keep the even powers of $\cos\phi'$, giving

$$\begin{aligned}
M_\ell &= \sigma_0 R^{\ell+2} \int_0^\pi \sin\theta'\, d\theta' \int_0^{2\pi} d\phi' \, (\cos\alpha \, \cos\theta') \left[(\cos\alpha \, \cos\theta')^2 + 3(\sin\alpha \, \sin\theta' \, \cos\phi')^2 \right] P_\ell(\cos\theta'),\\
&= \pi\sigma_0 R^{\ell+2} \cos\alpha \int_0^\pi \sin\theta'\, d\theta' \, \cos\theta' \, \left[2\cos^2\alpha \, \cos^2\theta' + 3\sin^2\alpha \, \sin^2\theta' \right] P_\ell(\cos\theta'),\\
&= \pi\sigma_0 R^{\ell+2} \cos\alpha \int_0^\pi \sin\theta'\, d\theta' \, \cos\theta' \, \left[(2\cos^2\alpha - 3\sin^2\alpha)\cos^2\theta' + 3\sin^2\alpha \right] P_\ell(\cos\theta'),\\
&= \pi\sigma_0 R^{\ell+2} \cos\alpha \int_0^\pi \sin\theta'\, d\theta' \left[(5\cos^3\alpha - 3\cos\alpha)\cos^3\theta' - (3\cos^3\alpha - 3\cos\alpha)\cos\theta' \right] P_\ell(\cos\theta'),
\end{aligned}$$

where in the 2nd step I used that the ϕ' integral of $\cos^2\phi'$ is π, in the 3rd and 4th steps I used a trig identities and collected terms. Since $P_1(\cos) = \cos$ and $P_3(\cos) = \frac{1}{2}(5\cos^3 - 3\cos)$, it follows that $\cos^3 = \frac{1}{5}(2P_3 + 3P_1)$. Plugging this in to the above expression for both $\cos\theta'$ and $\cos\alpha$ gives

$$\begin{aligned}
M_\ell &= \pi\sigma_0 R^{\ell+2} \int_0^\pi \sin\theta'\, d\theta' \Big[2P_3(\cos\alpha) \frac{1}{5} \left(2P_3(\cos\theta') + 3P_1(\cos\theta') \right) \\
&\qquad - \Big(\frac{3}{5}[2P_3(\cos\alpha) + 3P_1(\cos\alpha)] - 3P_1(\cos\alpha) \Big) P_1(\cos\theta') \Big] P_\ell(\cos\theta'),\\
&= \frac{2\pi}{5}\sigma_0 R^{\ell+2} \int_0^\pi \sin\theta'\, d\theta' \Big[2P_3(\cos\alpha) P_3(\cos\theta') \, P_\ell(\cos\theta') + 3P_1(\cos\alpha) P_1(\cos\theta') \, P_\ell(\cos\theta') \Big],\\
&= \frac{2\pi}{5}\sigma_0 R^{\ell+2} \frac{2}{2\ell+1} \Big[2P_3(\cos\alpha)\delta_{3,\ell} + 3P_1(\cos\alpha)\delta_{1,\ell} \Big] = \frac{4\pi}{5}\sigma_0 R^{\ell+2} \Big[\frac{2}{7} P_3(\cos\alpha)\delta_{3,\ell} + P_1(\cos\alpha)\delta_{1,\ell} \Big].
\end{aligned}$$

Therefore the multipole expansion is

$$V(r,\alpha) = \frac{1}{4\pi\epsilon_0} \sum_\ell \frac{M_\ell}{r^{\ell+1}} = \frac{\sigma_0}{5\epsilon_0} P_1(\cos\alpha) \frac{R^3}{r^2} + \frac{2\sigma_0}{35\epsilon_0} P_3(\cos\alpha) \frac{R^5}{r^4}.$$