Problem Set 1

Problems are worth one point each.

$\{\widehat{x}, \widehat{y}, \widehat{z}\}$ is the usual (orthonormal) Cartesian basis of a 3d vector space. Define a new basis $\{\widehat{x}, \widehat{y}', \widehat{z}'\}$ by

$$\widehat{x}' = \frac{1}{2}\left(-\widehat{x} - \widehat{y} + \sqrt{2}\widehat{z}\right), \qquad \widehat{y}' = \frac{1}{2}\left(\widehat{x} + \widehat{y} + \sqrt{2}\widehat{z}\right), \qquad \widehat{z}' = \frac{1}{2}\left(-\sqrt{2}\widehat{x} + \sqrt{2}\widehat{y}\right).$$

Problem 1. Compute $\widehat{x}' \cdot \widehat{x}'$, $\widehat{y}' \cdot \widehat{y}'$, and $\widehat{x}' \cdot \widehat{z}'$.

Solution:

$$\begin{aligned}
\widehat{x}' \cdot \widehat{x}' &= \frac{1}{2}\left(-\widehat{x} - \widehat{y} + \sqrt{2}\widehat{z}\right) \cdot \frac{1}{2}\left(-\widehat{x} - \widehat{y} + \sqrt{2}\widehat{z}\right) \\
&= \frac{1}{4}\left[\left(-\widehat{x} - \widehat{y} + \sqrt{2}\widehat{z}\right) \cdot (-\widehat{x}) + \left(-\widehat{x} - \widehat{y} + \sqrt{2}\widehat{z}\right) \cdot (-\widehat{y}) + \left(-\widehat{x} - \widehat{y} + \sqrt{2}\widehat{z}\right) \cdot \left(\sqrt{2}\widehat{z}\right)\right] \\
&= \frac{1}{4}\left[(-1)^2 + (-1)^2 + (\sqrt{2})^2\right] = 1.
\end{aligned}$$

We used that $\widehat{x} \cdot \widehat{x} = \widehat{y} \cdot \widehat{y} = \widehat{z} \cdot \widehat{z} = 1$ and $\widehat{x} \cdot \widehat{y} = \widehat{y} \cdot \widehat{z} = \widehat{z} \cdot \widehat{x} = 0$. Similarly,

$$\begin{aligned}
\widehat{y}' \cdot \widehat{y}' &= \frac{1}{2}\left(\widehat{x} + \widehat{y} + \sqrt{2}\widehat{z}\right) \cdot \frac{1}{2}\left(\widehat{x} + \widehat{y} + \sqrt{2}\widehat{z}\right) = \frac{1}{4}\left[(1)^2 + (1)^2 + (\sqrt{2})^2\right] = 1, \\
\widehat{x}' \cdot \widehat{z}' &= \frac{1}{2}\left(-\widehat{x} - \widehat{y} + \sqrt{2}\widehat{z}\right) \cdot \frac{1}{2}\left(-\sqrt{2}\widehat{x} + \sqrt{2}\widehat{y}\right) = \frac{1}{4}\left[(-1)(-\sqrt{2}) + (-1)(\sqrt{2}) + (\sqrt{2})(0)\right] = 0.
\end{aligned}$$

Problem 2. What is the norm (length) of $2\widehat{x}' + 3\widehat{y}'$?

Solution: $|2\widehat{x}' + 3\widehat{y}'| = \sqrt{(2\widehat{x}' + 3\widehat{y}') \cdot (2\widehat{x}' + 3\widehat{y}')} = \sqrt{4\widehat{x}' \cdot \widehat{x}' + 12\widehat{x}' \cdot \widehat{y}' + 9\widehat{y}' \cdot \widehat{y}'} = \sqrt{4 \cdot 1 + 12 \cdot 0 + 9 \cdot 1} = \sqrt{13}$, using the results of the last problem.

Problem 3. What is the angle between $\widehat{x}$ and $\widehat{x}'$?

Solution: Call the angle θ. Then $\cos\theta = (\widehat{x} \cdot \widehat{x}')/(|\widehat{x}| \cdot |\widehat{x}'|)$. Since $|\widehat{x}| = 1$ by definition, and $|\widehat{x}'| = 1$ by the first problem, we have $\cos\theta = \widehat{x} \cdot \widehat{x}' = \widehat{x} \cdot \frac{1}{2}\left(-\widehat{x} - \widehat{y} + \sqrt{2}\widehat{z}\right) = -\frac{1}{2}$, therefore $\theta = 2\pi/3$.

Problem 4. Compute $\widehat{x}' \times \widehat{y}'$ and $\widehat{y}' \times \widehat{z}'$.

Solution:

$$\widehat{x}' \times \widehat{y}' = \det\begin{pmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \left(-\tfrac{\sqrt{2}}{4} - \tfrac{\sqrt{2}}{4}\right)\widehat{x} + \left(\tfrac{\sqrt{2}}{4} + \tfrac{\sqrt{2}}{4}\right)\widehat{y} + \left(-\tfrac{1}{4} + \tfrac{1}{4}\right)\widehat{z} = \frac{1}{2}\left(-\sqrt{2}\widehat{x} + \sqrt{2}\widehat{y}\right) = \widehat{z}'.$$

$$\widehat{y}' \times \widehat{z}' = \det\begin{pmatrix} \widehat{x} & \widehat{y} & \widehat{z} \\ \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{pmatrix} = \left(0 - \tfrac{1}{2}\right)\widehat{x} + \left(-\tfrac{1}{2} + 0\right)\widehat{y} + \left(\tfrac{\sqrt{2}}{4} + \tfrac{\sqrt{2}}{4}\right)\widehat{z} = \frac{1}{2}(-\widehat{x} - \widehat{y} + \sqrt{2}\widehat{z}) = \widehat{x}'.$$

Problem 5. Do the two bases (in the orders given) have the same or opposite orientations?

Solution: They have the same orientations because $\widehat{x}'\cdot(\widehat{y}'\times\widehat{z}') = \widehat{x}'\cdot\widehat{x}' = 1 > 0$ by the results of problems 1 and 4.

Let $\vec{r} = x\widehat{x} + y\widehat{y} + z\widehat{z}$ be a general position vector, $r = |\vec{r}|$ its length, and $\widehat{r} = \vec{r}/r$ is the unit vector in the direction of $\vec{r}$. Compute the following in Cartesian coordinates.

Problem 6. $\vec{\nabla} r^n$ for any integer n.

Solution: Use shorthands $\partial_x = \partial/\partial x$, etc., $r^n = (\sqrt{\vec{r}\cdot\vec{r}})^n = (x^2+y^2+z^2)^{n/2}$. Then $\vec{\nabla} r^n = (\widehat{x}\partial_x + \widehat{y}\partial_y + \widehat{z}\partial_z)\,(x^2+y^2+z^2)^{n/2}$. Since $\partial_x(x^2+y^2+z^2)^{n/2} = nx(x^2+y^2+z^2)^{(n-2)/2} = nxr^{n-2}$, and similarly for ∂_y and ∂_z, thus $\vec{\nabla} r^n = (\widehat{x}nx + \widehat{y}ny + \widehat{z}nz)\,r^{n-2} = n\,r^{n-2}\,\vec{r} = n\,r^{n-1}\,\widehat{r}$.

Problem 7. $\vec{\nabla}\cdot(r^n\widehat{r})$.

Solution: $\vec{\nabla}\cdot(r^n\widehat{r}) = \vec{\nabla}\cdot(r^{n-1}\vec{r}) = \vec{\nabla}\cdot\left(r^{n-1}(x\widehat{x}+y\widehat{y}+z\widehat{z})\right) = \widehat{x}\cdot\vec{\nabla}(xr^{n-1}) + \widehat{y}\cdot\vec{\nabla}(yr^{n-1}) + \widehat{z}\cdot\vec{\nabla}(zr^{n-1})$. Now $\widehat{x}\cdot\vec{\nabla}(xr^{n-1}) = \widehat{x}\cdot\left(r^{n-1}\vec{\nabla}x + x\vec{\nabla}r^{n-1}\right) = \widehat{x}\cdot\left(r^{n-1}\widehat{x} + (n-1)xr^{n-2}\widehat{r}\right) = r^{n-1} + (n-1)r^{n-2}x\widehat{x}\cdot\widehat{r}$ where I used that $\vec{\nabla}x = \widehat{x}$ and the result of the last problem. And similarly for the y and z terms. So $\vec{\nabla}\cdot(r^n\widehat{r}) = r^{n-1} + (n-1)r^{n-2}x\widehat{x}\cdot\widehat{r} + r^{n-1} + (n-1)r^{n-2}y\widehat{y}\cdot\widehat{r} + r^{n-1} + (n-1)r^{n-2}z\widehat{z}\cdot\widehat{r} = 3r^{n-1} + (n-1)r^{n-2}(x\widehat{x}+y\widehat{y}+z\widehat{z})\cdot\widehat{r} = 3r^{n-1} + (n-1)r^{n-2}\vec{r}\cdot\widehat{r} = 3r^{n-1} + (n-1)r^{n-2}r = (n+2)r^{n-1}$.

Problem 8. $\vec{\nabla}\times(r^n\widehat{r})$.

Solution: Instead of computing directly as in the previous problem, use the identity $\vec{\nabla}\times(f\vec{A}) = (\vec{\nabla}f)\times\vec{A} + f\vec{\nabla}\times\vec{A}$ with $f = r^{n-1}$ and $\vec{A} = \vec{r}$. Since

$$\vec{\nabla}\times\vec{r} = \det\begin{pmatrix}\widehat{x} & \widehat{y} & \widehat{z}\\ \partial_x & \partial_y & \partial_z\\ x & y & z\end{pmatrix} = \widehat{x}(\partial_y z - \partial_z y) + \text{etc.} = 0, \tag{1}$$

and using problem 6, gives $\vec{\nabla}\times(r^n\widehat{r}) = (n-1)r^{n-3}\vec{r}\times\vec{r} = 0$, since $\vec{r}\times\vec{r} = 0$ by antisymmetry of the cross product.

Problem 9. $\nabla^2(r^n)$.

Solution: $\nabla^2(r^n) = \vec{\nabla}\cdot\vec{\nabla}(r^n) = \vec{\nabla}\cdot(nr^{n-1}\widehat{r}) = n(n+1)r^{n-2}$, using problems 6 & 7.

Problem 10. Prove $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$ for a general vector field $\vec{A}$.

Solution: In cartesian coordinates

$$\begin{aligned}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) &= \vec{\nabla}\times[\widehat{x}(\partial_y A_z - \partial_z A_y) + \widehat{y}(\partial_z A_x - \partial_x A_z) + \widehat{z}(\partial_x A_y - \partial_y A_x)]\\ &= \widehat{x}[\partial_y(\partial_x A_y - \partial_y A_x) - \partial_z(\partial_z A_x - \partial_x A_z)] + \widehat{y}[\partial_z(\partial_y A_z - \partial_z A_y) - \partial_x(\partial_x A_y - \partial_y A_x)]\\ &\quad + \widehat{z}[\partial_x(\partial_z A_x - \partial_x A_z) - \partial_y(\partial_y A_z - \partial_z A_y)].\end{aligned}$$

On the other hand:

$$\begin{aligned}\vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} &= \vec{\nabla}(\partial_x A_x + \partial_y A_y + \partial_z A_z) - (\partial_x^2 + \partial_y^2 + \partial_z^2)\vec{A}\\ &= \widehat{x}\partial_x(\partial_x A_x + \partial_y A_y + \partial_z A_z) + \widehat{y}\partial_y(\partial_x A_x + \partial_y A_y + \partial_z A_z) + \widehat{z}\partial_z(\partial_x A_x + \partial_y A_y + \partial_z A_z)\end{aligned}$$

$$
\begin{aligned}
&-\widehat{x}(\partial_x^2+\partial_y^2+\partial_z^2)A_x-\widehat{y}(\partial_x^2+\partial_y^2+\partial_z^2)A_y-\widehat{z}(\partial_x^2+\partial_y^2+\partial_z^2)A_z \\
=\,&\widehat{x}(\partial_x\partial_y A_y+\partial_x\partial_z A_z-\partial_y^2 A_x-\partial_z^2 A_x)+\widehat{y}(\partial_y\partial_x A_x+\partial_y\partial_z A_z-\partial_x^2 A_y-\partial_z^2 A_y) \\
&+\widehat{z}(\partial_z\partial_x A_x+\partial_z\partial_y A_y-\partial_x^2 A_z-\partial_y^2 A_z).
\end{aligned}
$$

Comparing, we see they are the same, term for term.