Lecture 10 Magnetostatics (Ch. 5)

- Magnetic field $\vec{B}(\vec{r})$ defined by noticing that a charge Q at $\vec{r}$ moving with velocity $\vec{v}$ experiences a force

$$\boxed{\vec{F} = Q\left(\vec{E}(\vec{r}) + \vec{v} \times \vec{B}(\vec{r})\right)}$$

Lorentz force law

(This defines both $\vec{E}$ & $\vec{B}$.)

⇒ Magnetic forces do no work on charged particles because the force is perpendicular to the velocity of the particle:

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = (Q\vec{v}\times\vec{B})\cdot(\vec{v}\,dt) = 0.$$

$\vec{F}_{mag}$ still accelerates particle, but only by changing direction, not speeding up or slowing down. So motion in magnetic fields tends to be in circles... (see Griffiths examples)

- Currents & current density

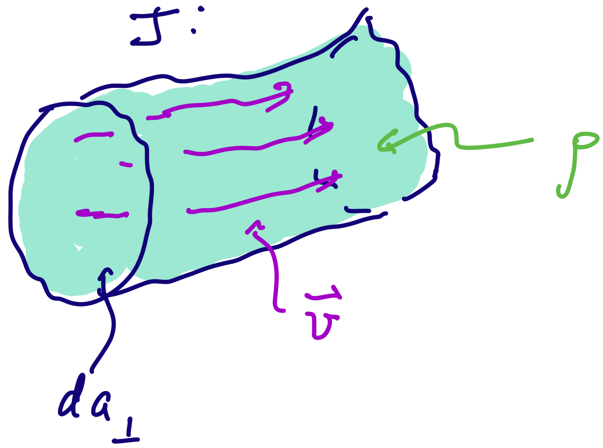
  - If some charges with density $\rho(\vec{r},t)$ are moving with velocities $\vec{v}(\vec{r},t)$, then the charge current density is

$$\vec{J}(\vec{r},t) \doteq \rho(\vec{r},t)\,\vec{v}(\vec{r},t)$$

$$= \frac{\text{charge}}{\text{volume}} \cdot \frac{\text{length}}{\text{time}} = \frac{\text{charge}}{\text{area} \cdot \text{time}}$$

$$= \frac{C}{s} \cdot \frac{1}{m^2} \qquad \frac{C}{s} = A \text{ (Ampere) "current"}$$

$\vec{J}$:



flow of charge through a surface

- In materials (e.g. conductors) can have different types of charges moving at different velocities simultaneously.
  E.g. electrons w/ $\rho_e(\vec{r},t)$, $\vec{v}_e(\vec{r},t)$
  and ions (nuclei) w/ $\rho_i(\vec{r},t)$, $\vec{v}_p(\vec{r},t)$.

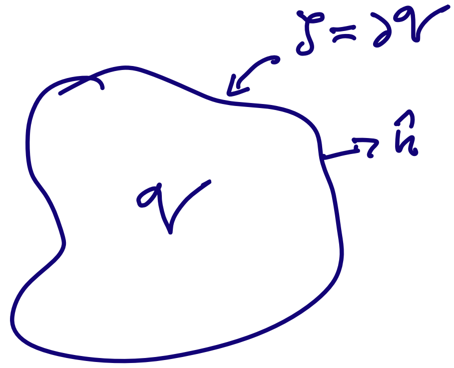
Then total charge & current densities are

$$\begin{cases} \rho = \rho_e + \rho_i \\ \vec{J} = \rho_e \vec{v}_e + \rho_i \vec{v}_i \end{cases}$$

So, in general $\vec{J} \neq \rho \vec{v}$!

E.g. in a metal $\rho_e = -\rho_i \Rightarrow \rho = 0$
but $\vec{v}_e \neq 0$ (electrons move) and $\vec{v}_i = 0$
(nuclei don't move) $\Rightarrow \vec{J} = \rho_e \vec{v}_e$.

- Charge is conserved: any net charge that flows into/out of a volume $\mathcal{V}$ will increase/decrease the charge inside:



Net charge flowing out of $\mathcal{V}$ per unit time:

$$\oint_S \vec{J} \cdot \hat{n}\, da = \oint_S \vec{J} \cdot d\vec{a}$$

Decrease in net charge inside $\mathcal{V}$ per unit time:

$$-\frac{d}{dt} \int_{\mathcal{V}} \rho \, d\tau = -\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} d\tau$$

Charge conservation:

$$-\int_{\mathcal{V}} d\tau \frac{\partial \rho}{\partial t} = \oint_S \vec{J} \cdot d\vec{a} = \int_{\mathcal{V}} d\tau \left(\vec{\nabla} \cdot \vec{J} \right) \qquad \forall \mathcal{V} \Rightarrow$$

$$\boxed{-\frac{\partial \rho}{\partial t} = \vec{\nabla}\cdot\vec{J}}$$

Charge conservation, "continuity equation"

- If you have a continuous, steady stream of moving charged particles, then $\vec{v}_e(\vec{r},t) = \vec{v}_e(\vec{r})$ & $\rho_e(\vec{r},t) = \rho_e(\vec{r})$

$$\Rightarrow \begin{cases} \vec{J}(\vec{r},t) = \vec{J}(\vec{r}) & \leftarrow \text{"steady" or "stationary" current} \\ \rho(\vec{r},t) = \rho(\vec{r}) & \leftarrow \text{"static" charge density.} \end{cases}$$

Steady currents and static charges, $\partial_t \vec{J} = \partial_t \rho = 0$, & charge conservation $\Rightarrow$

$$\boxed{\vec{\nabla}\cdot\vec{J} = 0}$$ for steady currents (& static charges)

So can't choose arbitrary steady $\vec{J}(\vec{r})$!

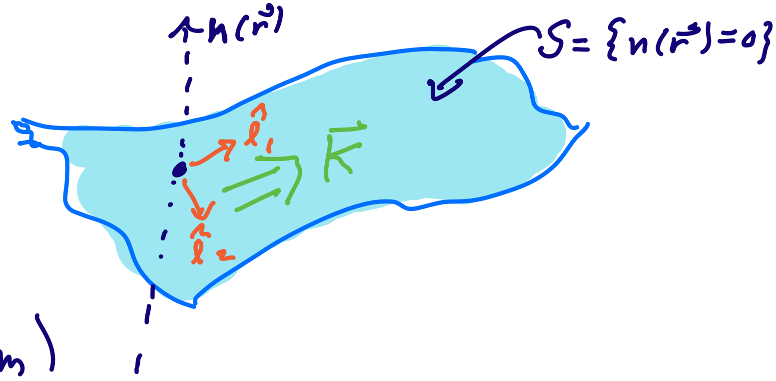
- Surface, line, & point current densities

- If moving charges confined to a surface, S, then

$$\vec{J}(\vec{r},t) = \vec{K}(\vec{r},t)\,\delta(n)$$ ← δ-fnc restricting to S

$\vec K$ = surface current density

$= \dfrac{\text{charge}}{\text{length}\cdot\text{time}}$ (A/m)



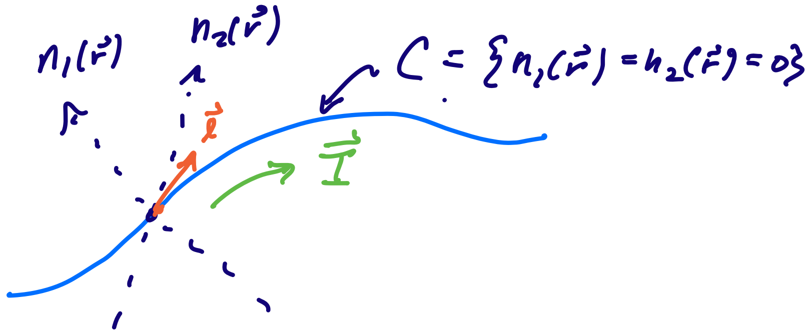
- If move charges confined to a curve, C, then

$$\vec J(\vec r, t) = \vec I(\vec r, t)\,\delta(n_1)\delta(n_2)$$ ← δ-fncs restricting to C

$\vec I$ = Line current density

= "current"

$= \dfrac{\text{charge}}{\text{time}}$ (A)



- For these to make sense (charge conservation) the charges must flow along (tangent to) the surface S or curve C:

$$\vec K \cdot \hat n = 0 \qquad \vec I \cdot \hat n_1 = \vec I \cdot \hat n_2 = 0$$

$$\Rightarrow \vec K = K_1 \hat l_1 + K_2 \hat l_2 \qquad \Rightarrow \vec I = I \hat l$$

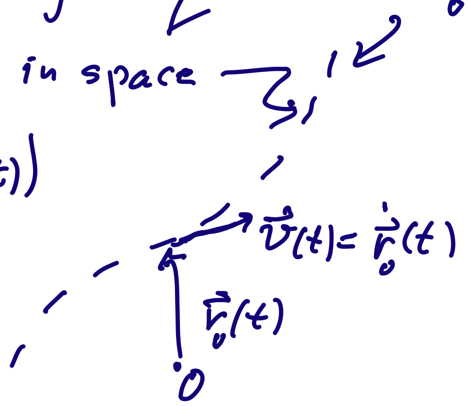
Steady current conservation, $\vec\nabla \cdot \vec J = 0$ implies

$$0 = \vec\nabla_\parallel \cdot \vec K = \partial_{l_1} K_1 + \partial_{l_2} K_2$$

$$0 = \vec{\nabla}_{\parallel} \cdot \vec{I} = \partial_\ell I \quad \Rightarrow \quad I = \text{constant}. \quad *$$

$*$ Steady line currents = constant currents down wires, are the main experimental current realization.

- For a moving point charge q at $\vec{r} = \vec{r}_0(t)$



$$\vec{J}(\vec{r}, t) = q\, \vec{v}(\vec{r}, t)\, \delta^3(\vec{r} - \vec{r}_0(t))$$

Even if velocity of particle is constant, $\vec{v} = \vec{v}(\vec{r})$, cannot have $\frac{\partial}{\partial t} \vec{J}(\vec{r}, t) = 0$!

So no such thing as a steady current of a particle.

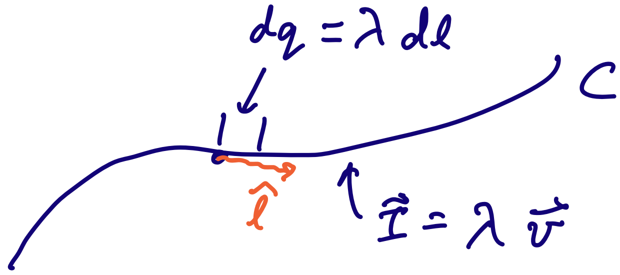
- point ~ line ~ surface ~ volume dictionary:

$$\text{charges:} \quad \sum_{i=1}^{n} (\cdot)\, q_i \sim \int_C (\cdot)\, \lambda\, d\ell \sim \int_S (\cdot)\, \sigma\, da \sim \int_V (\cdot)\, \rho\, d\tau$$

$$\text{currents:} \quad \sum_{i=1}^{n} (\cdot)\, q_i \vec{v}_i \sim \int_C (\cdot)\, \vec{I}\, d\ell \sim \int_S (\cdot)\, \vec{K}\, da \sim \int_V (\cdot)\, \vec{J}\, d\tau$$

$$\int_C (\cdot)\, \vec{I}\, d\ell = \int_C (\cdot)\, I\, d\vec{\ell} \qquad \int_S (\cdot)\, \vec{K}\, da \neq \int_S (\cdot)\, K\, d\vec{a}$$

– Magnetic force on a current-carrying wire

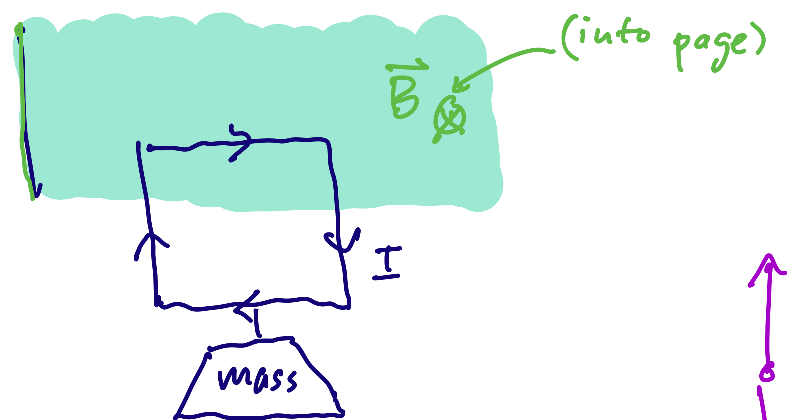


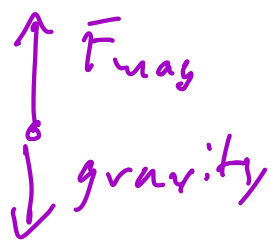
$\vec{I} = \lambda \vec{v} = \underbrace{I}_{\text{constant}} \hat{l}$

$$\vec{F}_{mag} = \int_C (\vec{v} \times \vec{B})\, dq = \int_C (\lambda \vec{v} \times \vec{B})\, dl$$

$$= \int_C (\vec{I} \times \vec{B})\, dl = I \int d\vec{l} \times \vec{B} .$$

- See example 5.3 of Griffiths:





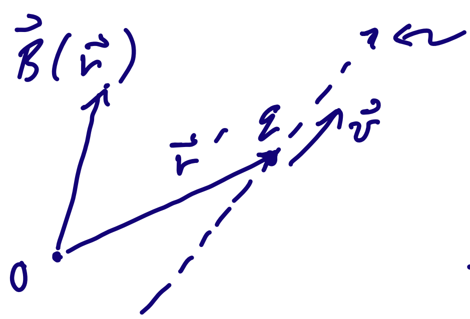
If $\vec{B}$ does no work, what lifts the mass??

- **Biot-Savart law**: moving charges source (create) magnetic fields:

It is found experimentally that a steadily moving charge q at position $\vec{r}\,'$ with velocity $\vec{v}$ creates a magnetic field at the point $\vec{r}$:

$$\boxed{\vec{B}(\vec{r},t) = \frac{\mu_0}{4\pi}\,\frac{q\,\vec{v}\times\hat{\mathscr{r}}}{\mathscr{r}^2}, \quad \vec{\mathscr{r}} = \vec{r}-\vec{r}\,'}$$

Biot-Savart Law



path of particle with constant velocity $\vec{v}$:

$\vec{r}\,' = \vec{r}_0 + \vec{v}\,t$

(Griffiths says this formula is <u>wrong</u> because it only applies if the particle is moving steadily (constant velocity) and if the velocity is small compared to the speed of light! So it is only an approximation. It becomes exact if $\vec{r}\,'(t) \to \vec{r}\,'(t_r)$ where "t_r" is the "retarded time" given implicitly by $t_r = t - \frac{1}{c}|\vec{r}-\vec{r}\,'(t_r)|$. In $c\to\infty$ limit, $t_r = t$.)

- Compare to Coulomb law:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q\,\hat{\imath}}{\imath^2}, \quad \vec{\imath} \doteq \vec{r} - \vec{r}\,'$$

(This also only applies if the particle's velocity is small compared to the speed of light!)

- Units & constants:

$\mu_0 \doteq 4\pi \times 10^{-7}\ N/A^2$ (exactly) $\left\{ \begin{array}{ll} N \doteq kg \cdot m^2/s^2 & \text{(Newton)} \\ A \doteq C/s & \text{(Ampere)} \end{array} \right.$

"permeability of free space" (definition = defines A→C!)

$[\vec{B}]$ = Tesla (T) or gauss

$T \doteq N/(A \cdot m)$ gauss $\doteq 10^{-4}\,T$

(1 T = huge ; 1 gauss = everyday)

Also: $\mu_0 \epsilon_0 = c^{-2}$ (exactly)

& $c \doteq 299\,792\,458$ m/s (exactly)

(definition of m)

- Recognize

$$\frac{q\vec{v}\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \int d\tau'\, \frac{q\vec{v}(\vec{r}\,')\,\delta^3(\vec{r}\,'-\vec{r}_0(t))\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \int d\tau'\,\frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

so Biot-Savart for general current density:

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau'\,\frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}}$$

Biot-Savart Law

This is exact for steady currents (no time-dependence).

For a steady line current $\vec{I}(\vec{r}\,') = I\,d\vec{l}\,'$,

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}\,'\times\hat{\mathscr{r}}}{\mathscr{r}^2}}$$

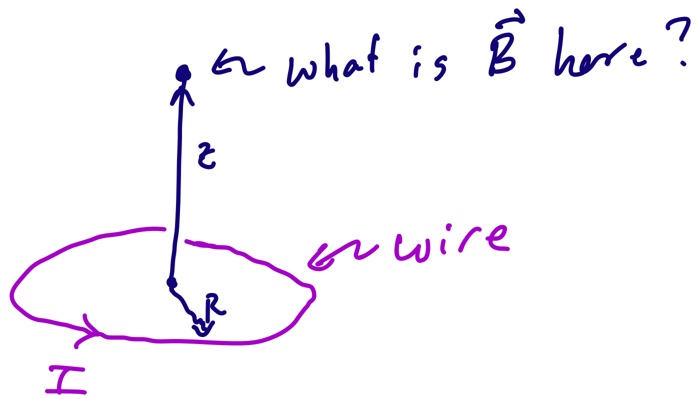
Biot-Savart Law

- There is something funny going on here: moving charges create $\vec{B}$-field, but motion is <u>relative</u>: I can go to an inertial frame where the charge is stationary. Does the $\vec{B}$-field go away? Yes!

Electric & magnetic fields are frame-dependent: different inertial observers will see different values of $\vec{E}$ & $\vec{B}$!

(It turns out (special relativity...) that $\vec{E}\cdot\vec{B}$ and $E^2 - \frac{1}{\epsilon_0\mu_0}B^2$ are frame-indep't.)

- See example 5.6 of Griffiths:



He does the Biot-Savart integral efficiently using the symmetries...

- <u>Differential equations satisfied by $\vec{B}$</u>

B-S: $$\vec{B} = \frac{\mu_0}{4\pi}\int d\tau' \, \frac{\vec{J}(\vec{r}\,')\times\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int d\tau' \; \vec{\nabla}_{\vec{r}}\cdot\left(\vec{J}(\vec{r}\,')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

PS1

$$= \frac{\mu_0}{4\pi}\int d\tau' \left\{ \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\overbrace{\left(\vec{\nabla}_{\vec r}\times \vec{J}(\vec r')\right)}^{0} - \vec{J}(\vec r')\cdot\overbrace{\left(\vec{\nabla}_{\vec r}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}^{0}\right\}$$

$$\therefore \quad \boxed{\vec{\nabla}\cdot\vec{B} = 0\,.}$$

$$\Rightarrow \quad \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int d\tau' \; \vec{\nabla}_{\vec r}\times\left(\vec{J}(\vec r')\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$= \frac{\mu_0}{4\pi}\int d\tau' \left\{ \vec{J}(\vec r')\overbrace{\left(\vec{\nabla}_{\vec r}\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)}^{4\pi\delta^3(\vec{\mathscr{r}})\ \text{(Lect 3)}} - \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\overbrace{\left(\vec{\nabla}_{\vec r}\cdot\vec{J}(\vec r')\right)}^{0} + \overbrace{\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\vec{\nabla}_{\vec r}\right)\vec{J}(\vec r')}^{0} - \underbrace{\left(\vec{J}(\vec r')\cdot\vec{\nabla}_{\vec r}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}}_{=0\ \text{because:}} \right\}$$

$$-\left(\vec{J}'\cdot\vec{\nabla}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = +\left(\vec{J}'\cdot\vec{\nabla}'\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$= \left(\vec{J}'\cdot\vec{\nabla}'\right)\frac{x-x'}{\mathscr{r}^3}\,\hat{x} + (\cdots y \,\&\, z \cdots)$$

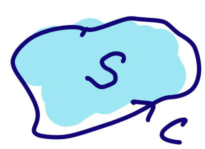
$$= \vec{\nabla}'\cdot\left[\frac{(x-x')}{\mathscr{r}^3}\vec{J}'\right]\hat{x} - \left(\frac{x-x'}{\mathscr{r}^3}\right)\hat{x}\,\underbrace{\vec{\nabla}'\cdot\vec{J}'}_{=0\ \text{steady current}} + (\cdots y \,\&\, z)$$

$$\therefore \quad \int_{\text{all space}} d\tau' \left(-\vec{J}'\cdot\vec{\nabla}\right)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \left(\oint_{\substack{\text{sphere}\\ \text{at } \infty}} \underbrace{d\vec{a}'\cdot\left(\frac{x-x'}{\mathscr{r}^3}\right)\vec{J}'}_{=0}\right)\hat{x} + \cdots$$

$$\therefore \quad \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}}$$

Ampere's law

Integrate over arbitrary open surface S and use Stoke's theorem to find

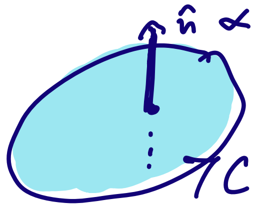


$$\oint_C \vec{B}\cdot d\vec{\ell} = \mu_0 \int_S \vec{J}\cdot d\vec{a} \doteq \mu_0 \underbrace{I_{enc}}_{\text{total current passing through } S}$$

for any C & any S such that $\partial S = C$

Ampere

(Remember RHR for Stokes theorem:



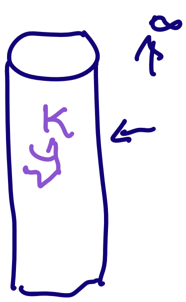
$\hat{n} \propto d\vec{a} \Rightarrow$ gives sign of I_{enc}.)

- Just as with the integral form of Gauss' law, $\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0} Q_{enc}$, the integral form of Ampere's law is useful when the problem has alot of symmetry:
  - infinite straight lines — Ex 5.7
  - infinite planes — Ex 5.8
  - infinite solenoids — Ex 5.9
  - circular solenoids ("toroids") — Ex 5.10

Ex 5.9 :

*

(prob 5.17 !)



constant surface current density

Ex 5.10

Electrostatics

static charges

$\rho = \rho(\vec{r})$

$\boxed{\vec{F}_{el} = Q\,\vec{E}}$

$\Leftarrow \boxed{\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}} \Rightarrow$

Magnetostatics

steady currents

$\vec{J} = \vec{J}(\vec{r})$ & $\vec{\nabla}\cdot\vec{J} = 0$

$\boxed{\vec{F}_{mag} = Q\,\vec{v}\times\vec{B}}$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \frac{\rho(\vec{r}')\hat{\mathscr{r}}}{\mathscr{r}^2} \quad \text{(Coul)} \qquad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau' \frac{\vec{J}(\vec{r}')\times\hat{\mathscr{r}}}{\mathscr{r}^2} \quad \text{(BS)}$$

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho \quad \text{(Gauss)} \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = 0 \quad \longleftarrow [\text{static}] \longrightarrow \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J} \quad \text{(Ampere)}$$

$$\oint_S \vec{E}\cdot d\vec{a} = \frac{1}{\epsilon_0}Q_{enc} \quad \text{(Gauss)} \qquad \oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \quad \text{(Ampere)}$$

- There is a kind of symmetry, interchanging
  "$\vec{E} \leftrightarrow \vec{B}$", "$\cdot \leftrightarrow \times$", "$\rho \leftrightarrow \vec{J}$"

  But this is only made precise (and useful) if understood in the context of special relativity.

- If there were magnetic charges ρ_B then would have (cgs units)

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho_E \qquad \vec{\nabla}\cdot\vec{B} = 4\pi\rho_B$$

$$\vec{\nabla}\times\vec{E} = \frac{4\pi}{c}\vec{J}_B \qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}_E$$

$$\vec{F} = Q_E\left(\vec{E} + \frac{1}{c}\vec{v}\times\vec{B}\right) + Q_M\left(\vec{B} + \frac{1}{c}\vec{v}\times\vec{E}\right)$$

Has exact symmetry under $E \leftrightarrow B$
"Electric-Magnetic duality".

- <u>Vector potential</u>

- Recall: $\vec{\nabla}\times\vec{E}=0 \Rightarrow$ exists V s.t. $\vec{E}=-\vec{\nabla}V$.
  - And V is not unique: $V' = V + \text{const}$ gives same $\vec{E}$.
  - And using V in Gauss' law $\vec{\nabla}\cdot\vec{E}=\frac{1}{\epsilon_0}\rho$ gives Poisson eqn $\nabla^2 V = -\frac{1}{\epsilon_0}\rho$.

- $\vec{\nabla}\cdot\vec{B}=0 \Rightarrow$ exists $\vec{A}$ s.t. $\boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$

  $\vec{A}$ = "magnetic vector potential".

- $\vec{A}$ is not unique: $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ for any function $\lambda(\vec{r})$ gives same $\vec{B}$:

$$\vec{\nabla}\times\vec{A}' = \vec{\nabla}\times\vec{A} + \underbrace{\vec{\nabla}\times\vec{\nabla}\lambda}_{0} = \vec{B}.$$

- Ampere's law:

$$\mu_0\vec{J} = \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$
$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$$

But: can alway choose $\vec{A}$ so that

$$\boxed{\vec{\nabla}\cdot\vec{A} = 0}$$ "Coulomb gauge"

Proof: If $\vec{\nabla}\cdot\vec{A} \neq 0$, define $\vec{A}' = \vec{A} + \vec{\nabla}\lambda$ such that $\nabla^2\lambda = -\vec{\nabla}\cdot\vec{A}$. (This is just Poisson's eqn for λ with "source" $\vec{\nabla}\cdot\vec{A}$, which we know has a solution.) Then

$$\vec{\nabla}\cdot\vec{A}' = \vec{\nabla}\cdot\vec{A} + \vec{\nabla}\cdot\vec{\nabla}\lambda = \vec{\nabla}\cdot\vec{A} + \nabla^2\lambda = \vec{\nabla}\cdot\vec{A} - \vec{\nabla}\cdot\vec{A} = 0.$$

- In Coulomb gauge, Ampere's law becomes

$$\boxed{\nabla^2\vec{A} = -\mu_0\vec{J}}$$

which is just a vector version of Poisson's equ, whose solution (component by component) via Coulomb's law* is

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau' \, \frac{\vec{J}(\vec{r}\,')}{\mathscr{r}}}$$ (assuming $\vec{J}\to 0$ @ ∞)

(* Compare: $\nabla^2 V = -\frac{1}{\epsilon_0}\rho \Leftrightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \, \frac{\rho(\vec{r}\,')}{\mathscr{r}}$)

- Not as useful as V in static situations...

But is useful for developing

• <u>Multipole expansion</u> (assume $\vec{J} \to 0$ @ ∞)

Use identity $\displaystyle \frac{1}{\mathscr{r}} = \sum_{l=0}^{\infty} \frac{(r')^l}{r^{l+1}} P_l(\underbrace{\cos\theta'}_{=\hat{r}\cdot\hat{r}'})$ $(r'<r)$

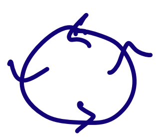
in expression for $\vec{A}$:

$$\boxed{\begin{aligned} \vec{A}(\vec{r}) &= \frac{\mu_0}{4\pi} \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \vec{M}_l(\hat{r}) \\ \vec{M}_l(\hat{r}) &\doteq \int d\tau' \, (r')^l \, \vec{J}(\vec{r}') \, P_l(\cos\theta') \end{aligned}}$$

l^{th} magnetic multipole.

- <u>Magnetic monopole</u> ($l=0$) <u>vanishes</u>:

$$\vec{M}_0(\hat{r}) \doteq \int d\tau' \, \vec{J}(\vec{r}') = 0$$

• Why? Intuitively, current is localized & conserved so must just circulate:  so integral must cancel.

<u>Proof</u>: (simple, but not in Griffiths):

By divergence theorem & $\vec{\nabla}' \cdot \vec{J} = 0$

$$0 = \int d\tau' \, \vec{\nabla}' \cdot (f \vec{J})$$

by div thm $= \oint_\infty = 0$ b/c $\vec{J}$ localized

$$= \int d\tau' \left(f \cancel{\vec{\nabla}' \cdot \vec{J}}^{\,0} + \vec{J} \cdot \vec{\nabla}' f \right)$$

$\therefore \quad 0 = \int d\tau' \, \vec{J} \cdot \vec{\nabla}' f$ for any function $f(\vec{r}')$!

Take $f = x_i'$ (one of the Cartesian coordinates)

Then $\vec{\nabla}' f = \hat{x}_i'$ so get

$$0 = \int d\tau \, J_i \, . \quad \therefore \quad \vec{M}_0(\hat{r}) = 0 . \quad \checkmark$$

- If write this for current loop C then

$$\int d\tau' \, \vec{J}(\vec{r}') \cdots \rightarrow \oint_C dl' \, \vec{I}(\vec{r}') \cdots = I \oint_C d\vec{l}' \cdots$$

so absence of magnetic monople becomes

$$0 = \int d\tau' \, \vec{J}(\vec{r}') = I \oint_C d\vec{l}' \Rightarrow \oint_C d\vec{l}' = 0 .$$

- Magnetic dipole ($l = 1$):

$$\vec{M}_1(\hat{r}) \doteq \int d\tau' \, r' \, P_1(\cos\theta') \, \vec{J}(\vec{r}')$$

$$= \int d\tau' \, r' \, \cos\theta' \, \vec{J}(\vec{r}')$$

$$= \int d\tau' \, r' \, (\hat{r}' \cdot \hat{r}) \, \vec{J}(\vec{r}')$$

$$= \int d\tau' \, (\hat{r} \cdot \vec{r}') \, \vec{J}(\vec{r}')$$

Consider i^{th} Cartesian component:

$$(M_1)_i = \hat{r} \cdot \int d\tau' \, \vec{r}\,' J_i(\vec{r}\,') \qquad \left(\hat{r} = \frac{\vec{r}}{r} = \frac{\sum x_j \hat{x}_j}{r}\right)$$

$$= \frac{1}{r} \sum_{j=1}^{3} x_j \int d\tau' \, x_j' J_i(\vec{r}\,')$$

Now use math lemma proved above with $f = x_i' x_j' \Rightarrow$

$$0 = \int d\tau' \, \vec{J}(\vec{r}\,') \cdot \vec{\nabla}(x_i' x_j')$$

$$= \int d\tau' \, \vec{J}(\vec{r}\,') \cdot [x_i' \hat{x}_j' + x_j' \hat{x}_i']$$

$$= \int d\tau' \, [x_i' J_j(\vec{r}\,') + x_j' J_i(\vec{r}\,')]$$

Use this to rewrite

$$\int d\tau' \, x_j' J_i(\vec{r}\,') = \frac{1}{2} \int d\tau' [x_j' J_i(\vec{r}\,') - x_i' J_j(\vec{r}\,')]$$

and plug into $(M_1)_i$ to get

$$(M_1)_i = \frac{-1}{2r} \sum_{j=1}^{3} x_j \int d\tau' (x_i' J_j(\vec{r}\,') - x_j' J_i(\vec{r}\,')]$$

$$= \frac{-1}{2r} \sum_{jk} x_j \epsilon_{ijk} \int d\tau' (\vec{r}\,' \times \vec{J}(\vec{r}\,'))_k$$

$$= -\frac{1}{2r} \left[\vec{r} \times \int d\tau' (\vec{r}\,' \times \vec{J}(\vec{r}\,'))\right]_k$$

$$\Rightarrow \quad \boxed{\vec{M}_1 = -\frac{1}{2}\,\hat{r} \times \int d\tau' \left(\vec{r}\,' \times \vec{J}(\vec{r}\,')\right)}$$

- It is conventional to define:

$$\boxed{\begin{aligned} \vec{\mathcal{M}}(\vec{r}\,') &\doteq \frac{1}{2}\,\vec{r}\,' \times \vec{J}(\vec{r}\,') \\ \vec{m} &\doteq \frac{1}{2}\int d\tau' \left(\vec{r}\,' \times \vec{J}(\vec{r}\,')\right) \end{aligned}}$$

"magnetic moment" density

"magnetic moment"

$(\vec{M}_1 = -\hat{r} \times \vec{m})$

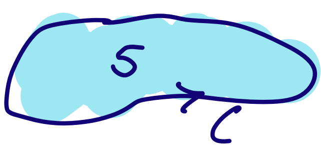
↳ analog of $\vec{p}$, electric dipole moment.)

- For current loop C: $\int d\tau' \vec{J}(\vec{r}\,') \to I \oint_C d\vec{\ell}\,'$

$$\vec{M}_1 = -\frac{1}{2}\,\hat{r} \times \left(I \oint_C \vec{r}\,' \times d\vec{\ell}\,'\right) = \frac{I}{2}\,\hat{r} \times \left(\oint_C d\vec{\ell}\,' \times \vec{r}\,'\right)$$

$$\vec{m} = \frac{I}{2} \oint_C \vec{r}\,' \times d\vec{\ell}\,' \doteq I \int_S d\vec{a}$$

where S in any surface s.t. $\partial S = C$



(Problem 1.61 of Griffiths)

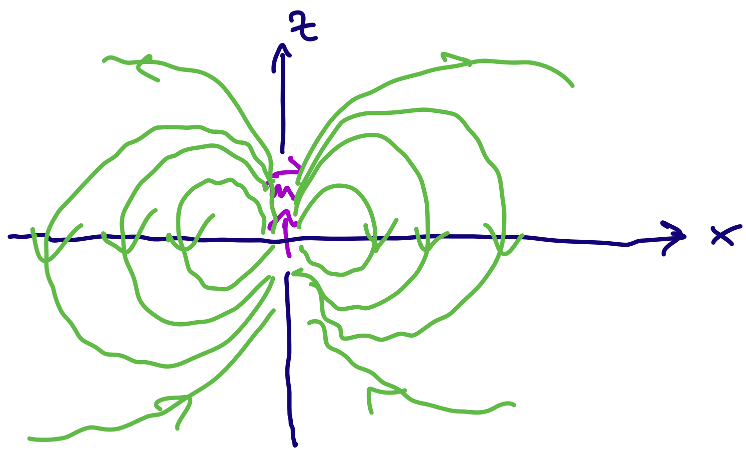
- "Pure" dipole = infinitesimal loop $\int_S d\vec{a} \to 0$ & $I \to \infty$

If $\vec{m} = m\,\hat{z}$, then

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{m \sin\theta}{r^2} \hat{\phi}$$

$$\vec{B}_{dip}(\vec{r}) = \vec{\nabla} \times \vec{A}_{dip} = \frac{\mu_0 m}{4\pi r^3} \left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta} \right)$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m} \right]$$



(Compare to electric dipole: $\vec{p} = p\,\hat{z}$:

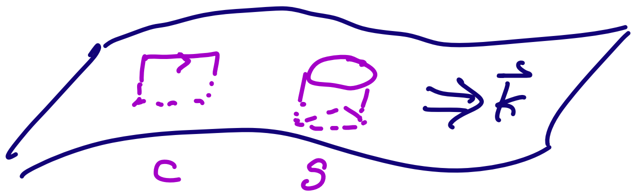
$$V_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{p \cos\theta}{r^2}$$

$$\vec{E}_{dip}(\vec{r}) = -\vec{\nabla} V_{dip} = \frac{p}{4\pi\epsilon_0} \frac{1}{r^3} \left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta} \right)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p} \right].$$)

- Boundary conditions on $\vec{B}$ & $\vec{A}$

Discontinuity at a surface current density $\vec{K}$



$$\vec{\nabla}\cdot\vec{B}=0 \Rightarrow \oint_S \vec{B}\cdot d\vec{a}=0$$

$$\lim_{S\to 0} \Rightarrow B^{\perp}_{above} = B^{\perp}_{below}$$

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J} \Rightarrow \oint_C \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$\lim_{C\to 0} \Rightarrow \vec{B}^{\parallel}_{above} - \vec{B}^{\parallel}_{below} = \mu_0 \vec{K}$$

Together $\Rightarrow$ $\boxed{\vec{B}_{above} - \vec{B}_{below} = \mu_0 (\vec{K}\times\hat{n})}$.

For $\vec{A}$: $\boxed{\frac{\partial}{\partial n}\vec{A}_{above} - \frac{\partial}{\partial n}\vec{A}_{below} = -\mu_0 \vec{K}}$

(Compare to $\vec{E}$-field. Discontinuity at surface charge density σ:

$$\vec{E}_{above} - \vec{E}_{below} = \frac{1}{\epsilon_0}\sigma\hat{n}$$

$$\frac{\partial}{\partial n}V_{above} - \frac{\partial}{\partial n}V_{below} = -\frac{1}{\epsilon_0}\sigma \quad .)$$