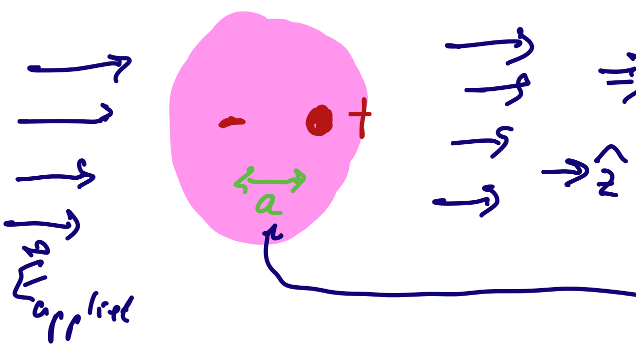
Lecture 9

Chapter 4: Electrostatics of insulators

- Insulator = Dielectric
  = material where charges are bound to each other in neutral units – atoms, molecules...
- So, unlike conductors, charges are not free to move large distances inside the material, but are confined to only atomic displacements.

} neutral spherically symmetric atom $\Rightarrow$ (multipole exp.) $\vec{E} = \frac{Q_{tot}}{4\pi\epsilon_0 r} = 0$

- Apply $\vec{E}$ field:



total potential

$$\Rightarrow V = -|E_{applied}|z + \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$$

$$= V_{applied} + V_{induced\ dipole}$$

induced dipole moment: $\vec{p} \sim e\vec{a} = \alpha \vec{E}_{applied}$

atomic polarizability

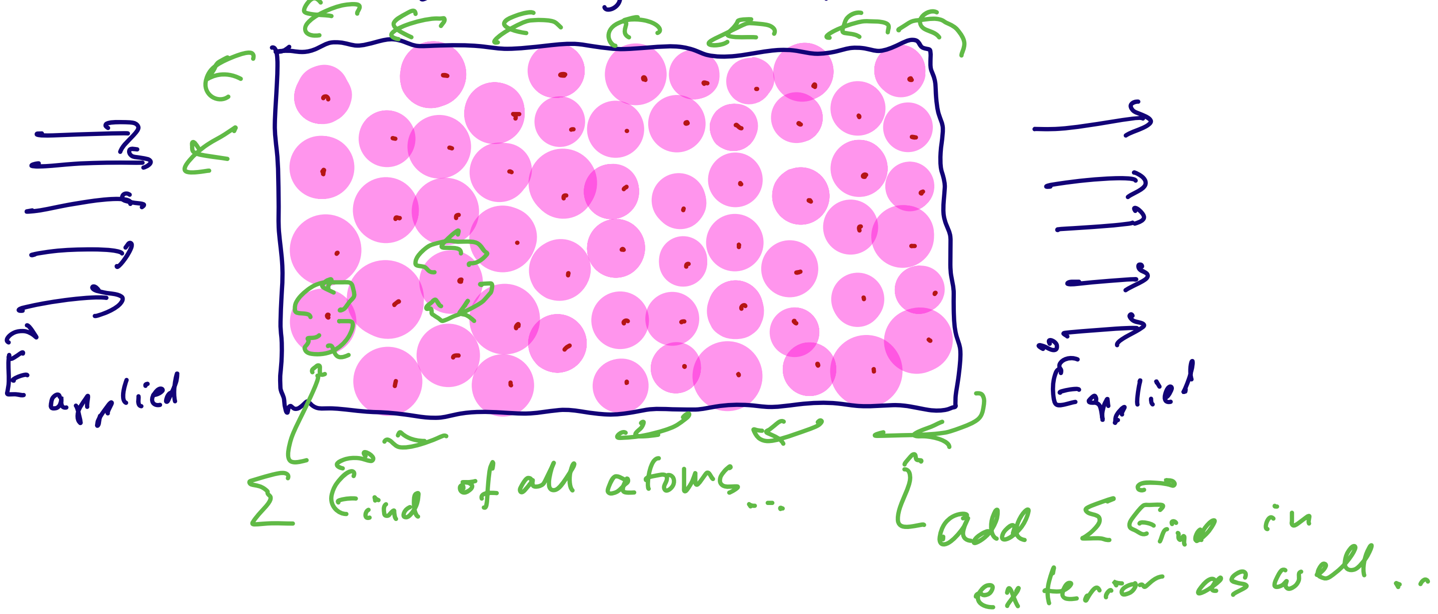
- More generally, in more complex materials $\vec{p} \not\propto \vec{E}$: write

$$p_i = \sum_{j=1}^{3} \underset{\uparrow}{\alpha_{ij}} E_j + \underbrace{\sum_{j,k=1}^{3} \beta_{ijk} E_j E_k + \cdots}_{\text{non-linear terms}}$$

polarizability tensor

(typically <u>very</u> small → ignore)

- Now apply to whole material, which is a collection of many atoms/molecules.



Very complicated! Solution: <u>average</u> total $\vec{E}$ field (applied + induced) over atomic scales to smooth out all the complicated variations. This is

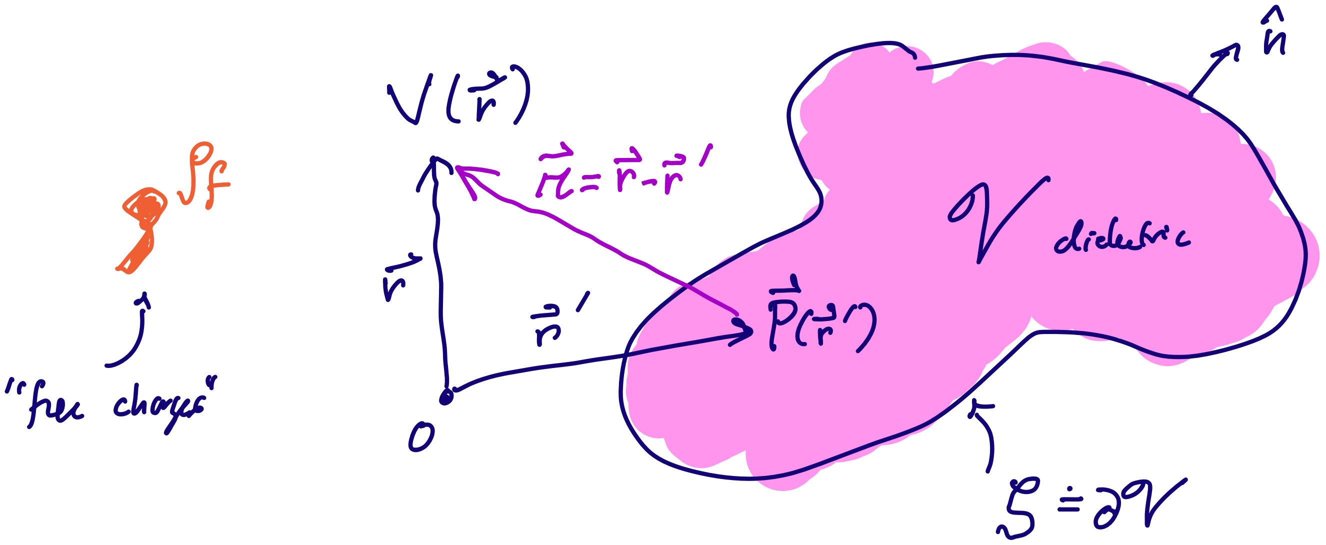
Fine as long as we only ask questions on length scales large compared to atomic scales.

There is an average induced dipole moment per unit volume

$$\vec{P}(\vec{r}\,') \doteq \frac{\vec{p}(\vec{r}\,')}{d\tau'} \quad \begin{matrix} \leftarrow \text{dipole moment of atom(s) at } \vec{r}\,' \\ \leftarrow \text{volume of atoms} \end{matrix}$$

$\uparrow$

"polarization vector".



$$V(\vec{r}) = \underbrace{V_{applied}(\vec{r})}_{\text{due to (distant) "free" charges } \rho_f(\vec{r})} + \frac{1}{4\pi\epsilon_0}\int_{\mathcal{V}} d\tau' \, \frac{\hat{\mathscr{r}} \cdot \vec{P}(\vec{r}\,')}{\mathscr{r}^2}$$

$$\Rightarrow V_{applied}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{S} d\tau' \, \frac{\rho_f(\vec{r}\,')}{\mathscr{r}}$$

all space (only $\neq 0$ where $\rho_f \neq 0$)

("Free" because they are not "bound" in the atoms of an insulating material.)

$$\therefore \boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \left(\frac{\rho_f(\vec{r}')}{\mathscr{r}} + \frac{\hat{\mathscr{r}}\cdot\vec{P}(\vec{r}')}{\mathscr{r}^2} \right)} \quad ⊛$$

Made integral over all space since $\vec{P}=0$ outside of material.

- It is very illuminating to rewrite the second term in ⊛ so that it looks like the first:

Recall $\dfrac{\hat{\mathscr{r}}}{\mathscr{r}^2} = \vec{\nabla}'\left(\dfrac{1}{\mathscr{r}}\right)$, so

$$\int d\tau' \frac{\hat{\mathscr{r}}\cdot\vec{P}}{\mathscr{r}^2} = \int d\tau' \, \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{\mathscr{r}}\right)$$

$$= \int d\tau' \left[\vec{\nabla}'\cdot\left(\frac{\vec{P}}{\mathscr{r}}\right) - \frac{1}{\mathscr{r}}\vec{\nabla}'\cdot\vec{P} \right]$$

$$= \cancelto{0}{\oint_\infty d\vec{a}'\cdot\frac{\vec{P}}{\mathscr{r}}} - \int d\tau' \frac{\vec{\nabla}'\cdot\vec{P}}{\mathscr{r}}$$

so $\circledast$ becomes

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \, \frac{\rho_f(\vec{r}') - \vec{\nabla}'\cdot\vec{P}(\vec{r}')}{\mathscr{r}}$$

- Define "bound charge density"

$$\boxed{\rho_b(\vec{r}) \doteq -\vec{\nabla}\cdot\vec{P}(\vec{r})}$$

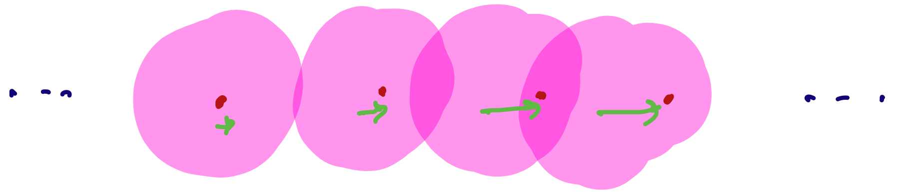
Then:

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d\tau' \, \frac{\rho_f(\vec{r}') + \rho_b(\vec{r}')}{\mathscr{r}}}$$

These are our key results for electrostatics of insulators. The rest consists of interpretation, reformulation, and approximation ...

- Interpretation

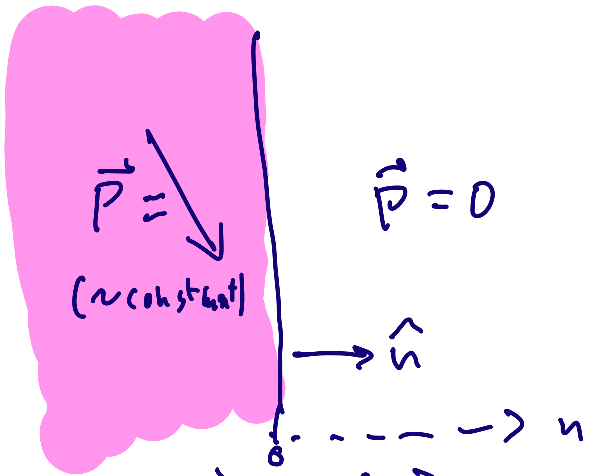
Even though an insulating material is charge-neutral, when polarized it can develop a charge density $\rho_b = -\vec{\nabla}\cdot\vec{P}$. This is due to a slight mismatch in the dipole moments of neighboring atoms so they don't exactly cancel:



Of course, overall it must cancel:

$$\int_V \rho_b \, d\tau' = 0 .$$

Generally ρ_b is very small, except right at the edges of the material.



$$\Rightarrow \rho_b = -\vec{\nabla} \cdot \vec{P} = \left(\vec{P}_{in} \cdot \hat{n} - \vec{P}_{out} \cdot \hat{n} \right) \delta(n)$$

$$\Rightarrow \boxed{\begin{aligned} \sigma_b &\doteq P^{\perp}_{inside} - P^{\perp}_{outside} \\ &= \hat{n} \cdot \vec{P} \Big|_{surface} \end{aligned}} \; \Bigg\} \text{ "bound surface charge"}$$

$\hat{n}$: outward-pointing unit normal

(Griffiths sometimes includes σ_b as a δ-function contribution to ρ_b, and sometimes separates it out...)

So, even if atomic induced dipole charges cancel (average to $\rho_b = 0$) in the interior, they will give rise to surface charge densities:

- <u>Reformulation</u>

Electrostatics eqns: $\vec{\nabla} \times \vec{E} = 0$ & $\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$

Rewrite Gauss: total charge

$$\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho = \rho_f + \rho_b = \rho_f - \vec{\nabla} \cdot \vec{P}$$

$$\Rightarrow \quad \vec{\nabla} \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f .$$

Define <u>electric displacement field</u>

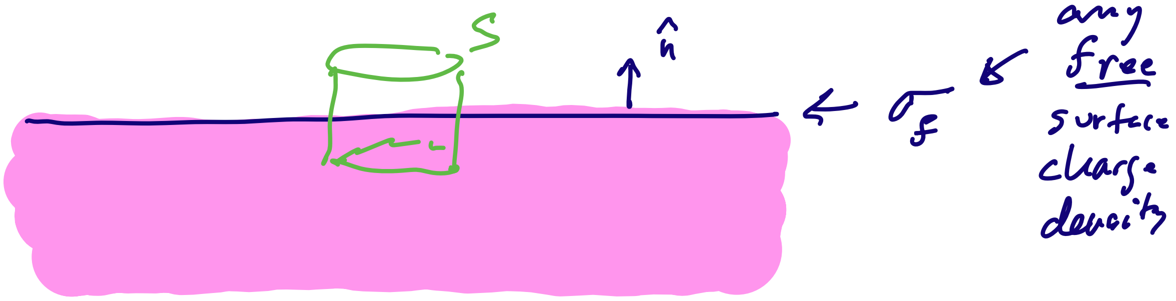
$$\boxed{\vec{D} \doteq \epsilon_0 \vec{E} + \vec{P}}$$

Then electrostatics eqns become:

$$\boxed{\vec{\nabla}\times\vec{E} = 0 \quad \& \quad \vec{\nabla}\cdot\vec{D} = \rho_f}$$

- These imply following boundary conditions at the surfaces of insulators:

$$\vec{\nabla}\cdot\vec{D} = \rho_f \Rightarrow \oint_S \vec{D}\cdot d\vec{a} = Q_{f\,\text{enclosed}}$$

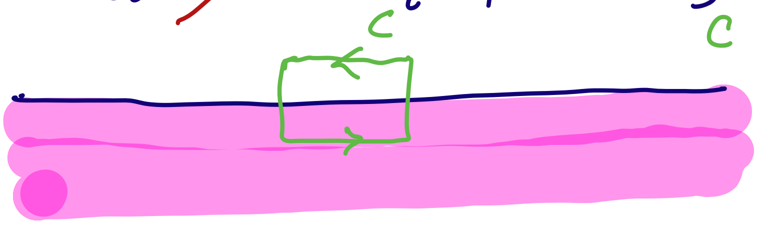


$$S\to 0 \Rightarrow \boxed{D^{\perp}_{\text{outside}} - D^{\perp}_{\text{inside}} = \sigma_f} \quad (D^{\perp} \doteq \vec{D}\cdot\hat{n}) \quad BC^{\perp}$$

usually $= \epsilon_0 E^{\perp}_{\text{outside}}$

(typically $\sigma_f = 0$).

$$\vec{\nabla}\times\vec{D} = \epsilon_0 \cancelto{0}{\vec{\nabla}\times\vec{E}} + \vec{\nabla}\times\vec{P} \Rightarrow \oint_C d\vec{l}\cdot\vec{D} = \oint_C d\vec{l}\cdot\vec{P}$$



$$C\to 0 \Rightarrow \boxed{\vec{D}^{\parallel}_{\text{outside}} - \vec{D}^{\parallel}_{\text{inside}} = \vec{P}^{\parallel}_{\text{outside}} - \vec{P}^{\parallel}_{\text{inside}}} \quad BC^{\parallel}$$

$$\Leftrightarrow \boxed{\vec{E}^{\parallel}_{\text{outside}} = \vec{E}^{\parallel}_{\text{inside}}}$$

- This reformulation is useful because it rewrites equations so that only the <u>free</u> charges appear. All the induced bound charges are implicitly packaged in terms of the polarization $\vec{P}(\vec{r})$.

- <u>Approximations</u>
  - So we need to understand the polarization of materials. The polarization is a <u>response</u> of the insulator to an electric field $\vec{E}$, and if $\vec{E}=0$ then $\vec{P}=0$.
  - If we know enough about the microscopic (quantum) physics of a given material, we could <u>compute</u> $\vec{P}(\vec{E})$. This is very difficult though (it is the subject of <u>condensed matter physics</u>...)
  - So we parameterize our ignorance by Taylor-expanding $\vec{P}(\vec{E})$:

$$P_i(\vec{r}) \doteq \sum_{j=1}^{3} \epsilon_0 \underbrace{(\chi_e)_{ij}(\vec{r})}_{} E_j(\vec{r}) + \sum_{j,k=1}^{3} \epsilon_0 (\varphi_e)_{ijk}(\vec{r}) E_j(\vec{r}) E_k(\vec{r})$$

$$+ \cdots,$$

$\underbrace{\qquad\qquad\qquad}_{\text{"linear response"}}$ $\underbrace{\qquad\qquad\qquad}_{\text{"non-linear response"}}$

↳ small for small $\vec{E}$...

"electric susceptibility tensor" = property of material

- Can measure $(\chi_e)_{ij}$, $(\varphi_e)_{ijk}$, ... experimentally. But we will make some approximations:

(1) Ignore non-linear terms "linear dielectrics"
(2) Assume $(\chi_e)_{ij} = \chi_e \delta_{ij}$ "isotropic " "
(3) Assume $\chi_e(\vec{r}) = \chi_e$ = constant "homogeneous " "

These are all good approximations for a large variety of materials.

- So from now on we will concentrate on "linear isotropic homogeneous dielectrics" (generally called just "linear dielectrics")

- Linear (isotropic homogeneous) dielectrics

$$\boxed{\vec{P}(\vec{r}) = \epsilon_0 \chi_e \vec{E}(\vec{r})}$$

χ_e = constant
= "electric susceptibility"

($\chi_e > 0$ in almost all materials.)

$$\Rightarrow \vec{D}(\vec{r}) = \underbrace{\epsilon_0 (1+\chi_e)} \vec{E}(\vec{r}) \qquad \vec{P}(\vec{r}) = \frac{\chi_e}{1+\chi_e} \vec{D}(\vec{r})$$

$$\doteq \epsilon \; \vec{E}(\vec{r}) \qquad \&$$

$\epsilon \doteq \epsilon_0 (1+\chi_e) \doteq$ "electric permittivity"

$(1+\chi_e) \doteq$ "dielectric constant" (dimensionless)

$$\Rightarrow \vec{\nabla} \cdot \vec{E} = \begin{cases} \frac{1}{\epsilon} \rho_f & \text{inside dielectric} \\ \frac{1}{\epsilon_0} \rho_f & \text{outside dielectric} \end{cases}$$

$$\Rightarrow \rho_b = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot \left(\frac{\chi_e}{1+\chi_e} \vec{D} \right) = -\frac{\chi_e}{1+\chi_e} \rho_f$$

$\therefore$ If no free charges inside material $\rho_f = 0$, (which is the usual situation) then

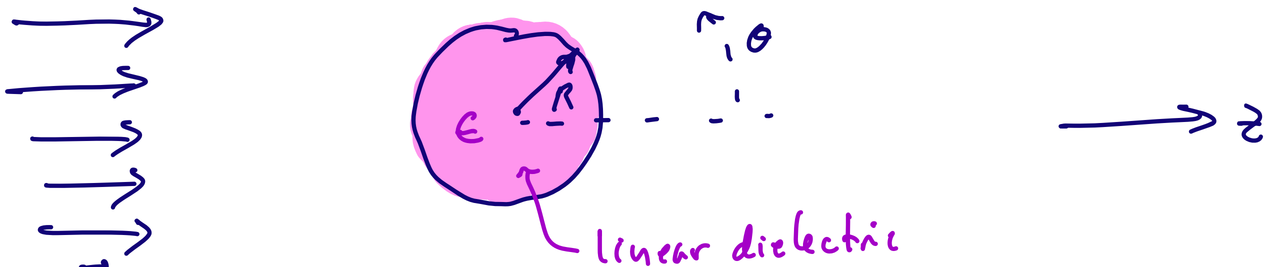
$$\boxed{\rho_b = 0 \quad \text{in interior of material}}$$

(There will still be surface $\sigma_b \neq 0$.)
This is very much like a conductor where $\rho = 0$ inside, and charge is only σ on bdry.

- Remember, though that χ_e or ϵ is constant inside the dielectric, but jumps discontinuously at its surface. So to solve $\{ \vec{\nabla}\cdot\vec{D} = \rho_f \, , \vec{\nabla}\times\vec{E} = 0, \ \vec{D} = \epsilon\vec{E} \}$ equations, you have to impose b.c.'s $BC^{\perp}$ & $BC^{\parallel}$ at surfaces of insulators.

- <u>Example boundary value problem</u>

(Ex 4.7 of Griffiths)



$\vec{E} = E_0 \hat{z}$

$\hookrightarrow$ applied $\vec{E}$-field = $\vec{E}$-field @ ∞.

What is $\vec{E}(\vec{r})$ for $r < R$?

Sol'n: Easiest in terms of pot'l $V(\vec{r})$

<u>Inside sphere</u>:

$$\left.\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{1}{\epsilon}\rho_f = 0 \\ \vec{\nabla}\times\vec{E} &= 0 \end{aligned}\right\} \Rightarrow \left\{ \begin{aligned} &\vec{E} = -\vec{\nabla} V_{in} \\ &\nabla^2 V_{in} = 0 \end{aligned}\right.$$

<u>Outside sphere</u>

$$\left.\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho_f = 0\right\} \Rightarrow \left\{ \vec{E} = -\vec{\nabla} V_{out} \right.$$

$$\vec{\nabla}\times\vec{E}=0 \qquad \int \qquad \wr\,\nabla^2 V_{out}=0$$

Boundary conditions:

- At $r\to\infty$: $\vec{E}=\epsilon_0\hat{z} \Rightarrow V_{out}=-E_0 z=-E_0 r\cos\theta$

$$\therefore \quad \boxed{V_{out}\to -E_0\, r\cos\theta \quad \text{for } r>>R}$$

- At $r=R$: $\begin{cases} D^{\perp}_{out}=D^{\perp}_{in} \Rightarrow \epsilon_0 E^{\perp}_{out}=\epsilon E^{\perp}_{in} \\ \vec{E}^{\parallel}_{out}=\vec{E}^{\parallel}_{in} \end{cases}$

$$\Rightarrow \left\{ \boxed{\begin{array}{ll} \epsilon_0 \dfrac{\partial V_{out}}{\partial r}=\epsilon\dfrac{\partial V_{in}}{\partial r} & \text{at } r=R \\ \\ V_{out}=V_{in} & \text{at } r=R \end{array}} \right.$$

- Solve Laplace's eqn by sep. o' variables:

$$\begin{cases} V_{in}(r,\theta)=\displaystyle\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\tilde{B}_\ell r^{-\ell-1}\right)P_\ell(\cos\theta) \\ V_{out}(r,\theta)=\displaystyle\sum_{\ell=0}^{\infty}\left(\tilde{A}_\ell r^\ell+B_\ell r^{-\ell-1}\right)P_\ell(\cos\theta) \end{cases}$$

- Impose boundary conditions.

• @ $r >> R$ $\quad V_{out}(r,\theta) = -E_0 r \cos\theta$

∴ $\tilde{A}_1 = -E_0$ & $\tilde{A}_\ell = 0 \quad \ell \neq 1$

• @ $r = 0$ $\quad V_{in}(0)$ = finite

∴ $\tilde{B}_\ell = 0 \quad \ell \geq 0$.

So

$$\begin{cases} V_{in}(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \\ V_{out}(r,\theta) = -E_0 r P_1(\cos\theta) + \sum_{\ell=0}^{\infty} B_\ell r^{-\ell-1} P_\ell(\cos\theta) \end{cases}$$

• @ $r = R$ $\quad V_{in} = V_{out}$ ⇒

$$\sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos\theta) = -E_0 r P_1(\cos\theta) + \sum_{\ell=0}^{\infty} B_\ell R^{-\ell-1} P_\ell(\cos\theta)$$

Equate $P_\ell(\cos\theta)$ terms (orthogonality) ⇒

$$\begin{cases} A_\ell R^\ell = B_\ell R^{-\ell-1} & \ell \neq 1 \\ A_1 R = -E_0 R + B_1 R^{-2} \end{cases}$$

#1

• @ $r=R$ $\quad \epsilon\, \partial_r V_{in} = \epsilon_0\, \partial_r V_{out} \implies$

$$\epsilon \sum_{l=0}^{\infty} l A_l R^{l-1} P_l(\cos\theta) =$$

$$= -\epsilon_0 E_0 P_1(\cos\theta) - \epsilon_0 \sum_{l=0}^{\infty} (l+1) B_l R^{-l-2} P_l(\cos\theta)$$

Equate P_l terms $\implies$

$$\begin{cases} \epsilon l A_l R^{l-1} = -\epsilon_0 (l+1) B_l R^{-l-2} & l \neq 1 \\ \epsilon A_1 = -\epsilon_0 E_0 - 2\epsilon_0 B_1 R^{-3} \end{cases} \quad ☆2$$

$$\implies \begin{cases} A_l = B_l = 0 \quad \text{for } l \neq 1 \\ A_1 = -\dfrac{3\epsilon_0}{\epsilon + 2\epsilon_0} E_0 \quad \& \quad B_1 = \dfrac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} R^3 E_0 \end{cases}$$

$$\implies V_{in}(r,\theta) = -\frac{3\epsilon_0 E_0}{\epsilon + 2\epsilon_0} r\cos\theta = -\frac{3\epsilon_0 E_0}{\epsilon + 2\epsilon_0} z$$

$$\implies \vec{E}_{in}(r,\theta) = -\vec{\nabla} V_{in} = \frac{3\epsilon_0 E_0}{\epsilon + 2\epsilon_0} \hat{z}$$

$$= \frac{3\epsilon_0}{\epsilon + 2\epsilon_0} \cdot \vec{E}_0 \qquad \text{uniform!}$$

- Other important topic covered in Griffiths Ch. 4:

§4.1.3 : forces on dipoles:

$$\left\{\begin{array}{l} \vec{N} = \vec{p} \times \vec{E} \quad \text{(torque about center)} \\ \vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} \quad \Rightarrow \mathcal{U} = -\vec{p}\cdot\vec{E} \quad \text{(pot'l energy)} \end{array}\right.$$

§4.4.3 : energy in dielectrics

$$W = \frac{1}{2}\int d\tau\, \vec{D}\cdot\vec{E}$$

§4.4.4 : forces on dielectrics ...

Many classic examples & problems ...