Lecture 8a   Worked problem

- Consider a thin sphere of radius R and surface charge density $\sigma = \sigma_0 \cos\theta$ in spherical coordinates. Find the multipole expansion of $V(\vec{r})$ for $r \gg R$.

- The multipole expansion is

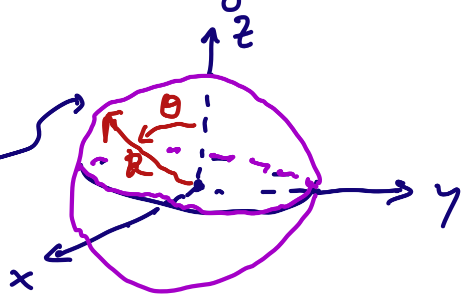
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{M_\ell(\hat{r})}{r^{\ell+1}} \quad \text{with}$$

$$M_\ell(\hat{r}) \doteq \int d\tau' \rho(\vec{r}\,') P_\ell(\cos\theta') \cdot (r')^\ell$$

where ρ is the (volume) charge density and $\cos\theta' = \hat{r} \cdot \hat{r}'$ is the angle between $\vec{r}$ and $\vec{r}\,'$.

- In this problem we are give a $\underline{\text{surface}}$ charge distribution:

$\sigma = \sigma_0 \cos\alpha$

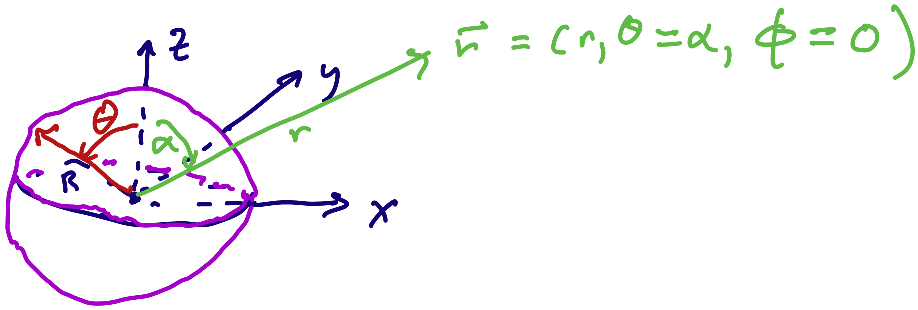


So we can write the volume charge density as $\rho(\vec{r}) = \sigma\, \delta(r-R)$

$$= \sigma_0 \cos\theta\, \delta(r-R)$$

in the spherical coordinates shown: $\vec{r} = (r, \theta, \phi)$.

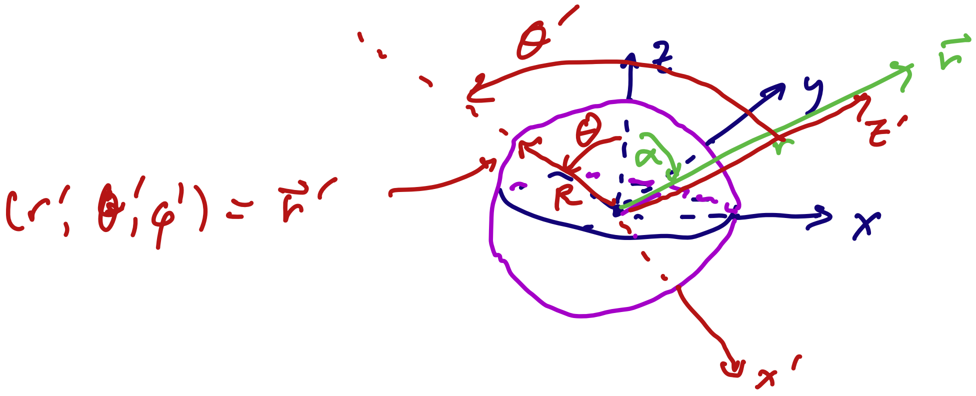
- We want to find $V(\vec{r})$



I have set $\phi = 0$ for $\vec{r}$ without loss of generality since the problem is rotationally symmetric around the z-axis. Note: I have renamed the θ-coordinate of $\vec{r}$ to $\theta = \alpha$ since I already used "θ" to denote a position on the spherical charge distribution.

- Now we need to choose a coordinate system for the $\vec{r}\,'$ coordinates that we integrate over to find $M_\ell(\hat{r})$.

The trick is to choose the z'-axis aligned along $\vec{r}$, and the x'-axis to be in the x-z plane:



$$\begin{cases} z = z'\cos\alpha - x'\sin\alpha \\ x = z'\sin\alpha + x'\cos\alpha \\ y = y' \end{cases}$$

(so y'-axis = y-axis). Then θ' is the angle between $\vec{r}$ and $\vec{r}'$, so we can use the definition of the multipole as is:

$$M_\ell(\hat{r}) = \int d\tau'\, \rho(\vec{r}\,')\cdot P_\ell(\cos\theta')\cdot (r')^\ell$$

$$= \int_0^\infty (r')^2 dr' \int_0^\pi \sin\theta'\, d\theta' \int_0^{2\pi} d\phi' \,\cdot$$

$$\sigma_0 \cdot \cos\theta \cdot \delta(r'-R)\cdot (r')^\ell P_\ell(\cos\theta')$$

- But we now need to write θ in terms of θ':

$$\cos\theta = \hat{r}'\cdot\hat{z} = \hat{r}'\cdot(\cos\alpha\, \hat{z}' - \sin\alpha\, \hat{x}')$$

$$= \cos\alpha\,(\hat{r}'\cdot\hat{z}') - \sin\alpha\,(\hat{r}'\cdot\hat{x}')$$

$$= \cos\alpha\,(\cos\theta') - \sin\alpha\,(\sin\theta'\cos\phi')$$

So we have

$$M_\ell(\hat{r}) = R^{\ell+2}\sigma_0 \int_0^{\pi}\sin\theta' d\theta' \int_0^{2\pi} d\phi' \cdot$$

$$\cdot\left[\cos\alpha\cos\theta' - \sin\alpha\sin\theta'\cos\phi'\right]\cdot P_\ell(\cos\theta')$$

- The ϕ' integral of the first term $= 2\pi$ and $=0$ for the second term, so

$$M_\ell(\hat{r}) = R^{\ell+2}\sigma_0\, 2\pi\cdot\cos\alpha \int_0^{\pi}\sin\theta' d\theta' \cos\theta' P_\ell(\cos\theta')$$

$$= R^{\ell+2}\sigma_0\, 2\pi\cos\alpha \int_0^{\pi}\sin\theta' d\theta' P_1(\cos\theta') P_\ell(\cos\theta')$$

$$= R^{\ell+2}\sigma_0\, 2\pi\cos\alpha\cdot\frac{2}{2\ell+1}\cdot\delta_{\ell,1}$$

$$\boxed{M_\ell(\hat{r}) = \begin{cases} \frac{4\pi}{3}R^3\sigma_0\cos\alpha & \ell=1 \\ 0 & \text{otherwise}\end{cases}}$$

where I used the orthogonality

relation for Legendre polynomials

$$\int_0^\pi \sin\theta \, d\theta \, P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1} \delta_{\ell,\ell'} .$$

- So only the dipole term survives! Using $\cos\alpha = \hat{z}\cdot\hat{r}$ we can write

$$M_1(\hat{r}) = \frac{4\pi}{3} R^3 \sigma_0 \, \hat{z}\cdot\hat{r} \doteq \vec{p}\cdot\hat{r} , \text{ so}$$

our final answer is:

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2} \quad \text{w/} \quad \vec{p} = \frac{4\pi}{3} R^3 \sigma_0 \, \hat{z}} .$$