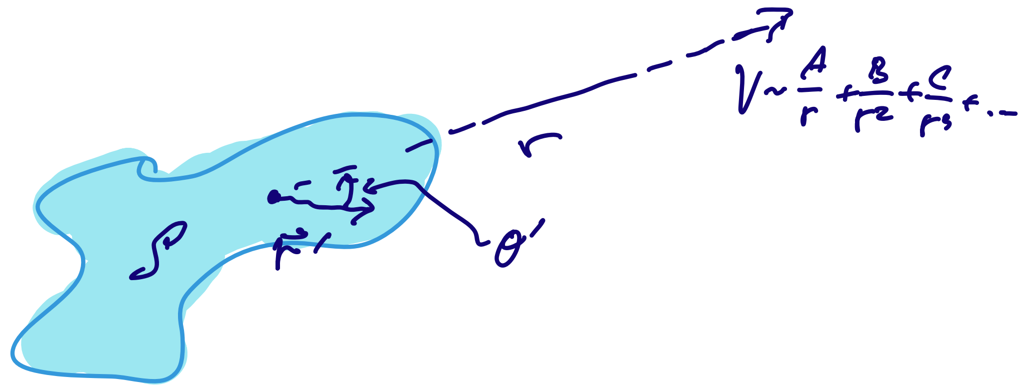
Lecture 8

Section 3.4 Multipole expansion

- Systematic expansion of field of a localized charge distribution in powers of $1/r$ where r is the distance from the charges:



- Coulomb's law:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int d\tau' \, \frac{\rho(\vec{r}')}{\mathscr{r}}$$

$\mathscr{r} = |\vec{r} - \vec{r}'|$ & $r \gg r'$, so rewrite

$$\frac{1}{\mathscr{r}} = \left(r^2 + (r')^2 - 2rr'\cos\theta'\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 - 2\frac{r'}{r}\cos\theta' + \left(\frac{r'}{r}\right)^2\right)^{-1/2}$$

small,

define $\quad \epsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta'\right) \ll 1$

$$\frac{1}{\imath} = \frac{1}{r}(1+\epsilon)^{-1/2}$$

$$= \frac{1}{r}\left(1 + \frac{\left(-\frac{1}{2}\right)}{1!}\epsilon + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\epsilon^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\epsilon^3 + \cdots\right)$$

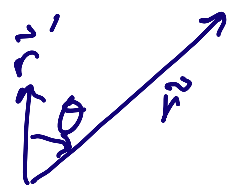
$$= \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

Plug in ϵ ... find:

$$\frac{1}{\imath} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

Proof:

$$\frac{1}{\imath} = \frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{\sqrt{r^2+(r')^2-2rr'\cos\theta}}$$



This is a function $V(r,\theta)$ which satisfies Laplace's eqn for $r>0$. Therefore it can be written as

$$\frac{1}{\mathscr{r}} = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1} \right) P_\ell(\cos\theta) \qquad (*)$$

by separation of variables in spherical coordinates for some values of the constants $\{A_\ell, B_\ell\}$.

We can determine these constants by looking at $\theta = 0$ for $r > r'$:

LHS (*):
$$\left.\frac{1}{\mathscr{r}}\right|_{\theta=0} = \frac{1}{\sqrt{r^2 + (r')^2 - 2rr'}} = \frac{1}{|r - r'|}$$

$$= \frac{1}{r - r'} = \frac{1}{r} \frac{1}{\left(1 - \frac{r'}{r}\right)}$$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell . \qquad (L)$$

$$\text{RHS } (*) \Big|_{\theta=0} = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1} \right) \overset{1}{\cancel{P_\ell(1)}}$$

$$= \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^\ell} + \sum_{\ell=0}^{\infty} A_\ell r^\ell \qquad (R)$$

Comparing (L) = (R) $\Rightarrow$

$$A_\ell = 0 \quad \forall \ell, \qquad B_\ell = (r')^\ell .$$

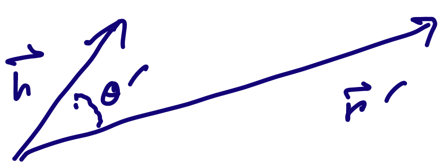
$$\therefore (\star) \Rightarrow \frac{1}{\mathscr{r}} = \sum_{\ell=0}^{\infty} (r')^\ell r^{-\ell-1} P_\ell(\cos\theta)$$

$$\boxed{= \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell P_\ell(\cos\theta). \quad \checkmark}$$

- Plugging $(\star)$ into Coulomb's law gives

$$\boxed{\begin{aligned} V(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \frac{M_\ell(\hat{r})}{r^{\ell+1}} \\ M_\ell(\hat{r}) &\doteq \int d\tau' \, (r')^\ell P_\ell(\cos\theta') \, \rho(\vec{r}\,') \end{aligned}}$$

"Multipole expansion"



$\Longleftrightarrow \quad \cos\theta' \doteq \hat{r} \cdot \hat{r}'$

so $M_\ell(\hat{r})$ is function of $\hat{r}$.

Since $P_\ell(\cos\theta')$ is degree-ℓ polynomial

$\Rightarrow$ can write

$$M_\ell(\hat{r}) = \sum_{i_1,\dots,i_\ell=1}^{3} \underbrace{M_{i_1\dots i_\ell}}_{\text{"Multipole moments"}} (\hat{r})_{i_1}(\hat{r})_{i_2}\cdots(\hat{r})_{i_\ell}$$

* $\underline{\ell=0}$ $\quad M \doteq Q \quad$ "monopole" moment (scalar)

* $\underline{\ell=1}$ $\quad M_i \doteq \vec{p} \quad$ "dipole" moment (vector)

$\underline{\ell=2}$ $\quad M_{ij} \doteq Q_{ij} \quad$ "quadrupole" moment (rk-2 tensor)

$\underline{\ell=3}$ $\quad M_{ijk} \quad$ "octupole" moment (rk-3 tensor)

$\quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots$

* We will focus almost exclusively on the 2 leading terms: monopole & dipole.

Monopole moment:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)$$

w/ $$Q \doteq M_0(\hat{r}) = \int d\tau' \, \cancelto{1}{P_0(\cos\theta')} \, \rho(\vec{r}\,')$$

$$= \int d\tau' \rho(\vec{r}\,') = \text{total charge}. \checkmark$$

Dipole moment:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{\hat{r}\cdot\vec{p}}{r^2}\right) + \mathcal{O}\left(\frac{1}{r^3}\right)$$

w/ $$\hat{r}\cdot\vec{p} \doteq M_1(\hat{r}) = \int d\tau' \cdot r' \cdot \cancelto{\cos\theta'}{P_1(\cos\theta')} \, \rho(\vec{r}\,')$$

$$= \int d\tau' \cdot r' \cos\theta' \cdot \rho(\vec{r}\,')$$

$$= \int d\tau' \, (\hat{r}\cdot\vec{r}\,') \cdot \rho(\vec{r}\,')$$

$$= \hat{r}\cdot\left(\int d\tau' \, \vec{r}\,' \rho(\vec{r}\,')\right)$$

$$\Rightarrow \boxed{\vec{p} \doteq \int d\tau' \, \vec{r}\,' \rho(\vec{r}\,')}$$

$$= \sum_i \vec{r}_i\, q_i$$ ← "charges weighted by their positions" (for point charges).

- Dependence on choice of origin: In general the multipole moments change if you change the origin of your coordinate system:

If you shift origin to new point $\vec{a}$ (in old coordinate system) then a moment shifts by amount proportional to the lower moments:

monopole: $M \doteq Q \longrightarrow Q$ (unchanged)

dipole: $M_i \doteq \vec{p} \longrightarrow \vec{p} - Q\vec{a}$

quadrupole: $M_{ij} \longrightarrow M_{ij} - a_i p_j - p_i a_j + a_i a_j Q$

$\vdots \qquad \vdots \qquad \vdots$

- Dipole $\vec{E}$ field:

$$V_{dip}(r,\theta) = \frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

where we have taken $\vec{p} \propto \hat{z}$, i.e. put the $+z$-axis along $\vec{p}$. Then

$$\boxed{\begin{aligned}\vec{E}_{dip}(r,\theta) &= -\vec{\nabla} V_{dip} \\ &= \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right) - \frac{1}{3\epsilon_0}\vec{p}\,\delta^3(\vec{r}) \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right] - \underbrace{\frac{1}{3\epsilon_0}\vec{p}\,\delta^3(\vec{r})}\end{aligned}}$$

(See problems 3.41 + 3.42 Griffiths)