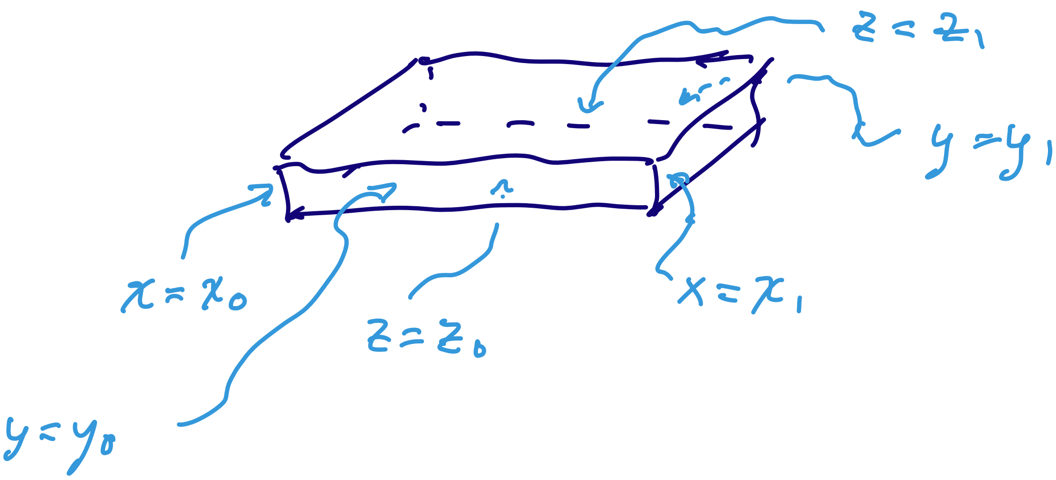
Lecture 7

3.3 Separation of variables

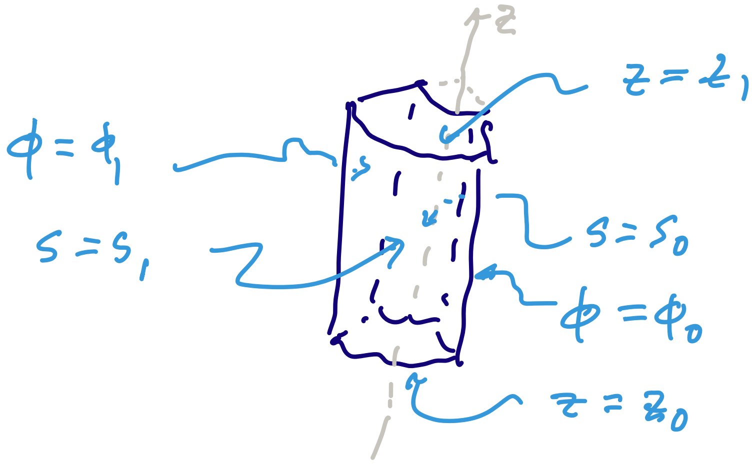
- Systematic method for finding series solutions of Laplace's equation when the boundary conditions are on surfaces described by constant values of coordinates in some coordinate system:

<u>Cartesian</u>: boundaries at $x = \text{constant}$ or $y = \text{constant}$ or $z = \text{constant}$:

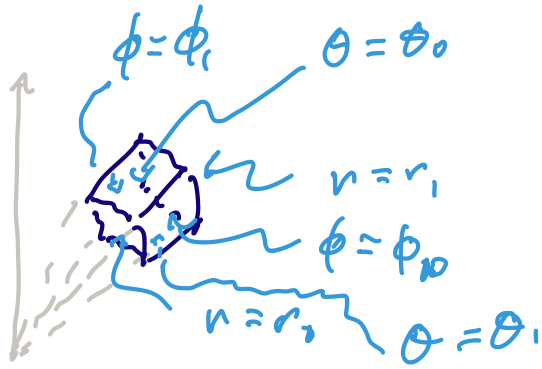


So, good in "rectangular" domains.
Note that x_i, y_i, z_i could be $\pm\infty$.

<u>Cylindrical</u>: bndrs @ $s = \text{const}$, $\phi = \text{const}$, $z = \text{const}$



Spherical: bdrs @ $r=\text{const}$, $\theta=\text{const}$, $\phi=\text{const}$



- Idea: look for solutions of $\nabla^2 V = 0$ such that:

cartesian $\quad V(x,y,z) = X(x)\cdot Y(y)\cdot Z(z)$

cylind. $\quad V(s,\phi,z) = S(s)\cdot \Phi(\phi)\cdot Z(z)$

spher. $\quad V(r,\theta,\phi) = R(r)\cdot \Theta(\theta)\cdot \Phi(\phi)$

Then find that $\nabla^2 V = 0 \Rightarrow$ 3 separate ordinary differential eqns for the factor functions (X, Y, Z) etc., each depending on a real constant (k_x^2, k_y^2, k_z^2)

These ordinary differential eqns can be solved once & for all & get a family of "special functions" e.g.

* Cartesian:

$$X_k \sim e^{\pm ikx} \text{ or } e^{\pm kx} \approx \begin{cases} \sin(kx) & \text{or } \sinh(kx) \\ \cos(kx) & \cosh(kx) \end{cases}$$

& same for Y_k, Z_k.

Cylindrical:

$S_k \sim$ Bessel functions J_k, Y_k

$\Phi_k \sim e^{\pm ik\phi}$ or $e^{\pm k\phi}$

$Z_k \sim e^{\pm ikz}$ or $e^{\pm kz}$

* Spherical:

$R_k \sim r^{l_+}, r^{l_-}$ $\qquad l_\pm = \dfrac{-1 \pm \sqrt{1+4k^2}}{2}$

$\Theta_k \sim$ (associated) Legendre polynomials $P_l^m(\cos\theta)$

$\Phi_k \sim e^{\pm ik\phi}$ or $e^{\pm k\phi}$

(* We'll just concentrate on cartesian & spherical in this course.)

- "Special functions" have very nice properties
  orthogonality & completeness

If $N_k(x)$ is a set of special functions, then, heuristically,

orthogonality:

$$\int dx\, N_k(x) N_{k'}(x) = \delta_{k,k'} \qquad \text{ON}$$

completeness:

$$\sum_k N_k(x) N_k(x') = \delta(x-x') \qquad \text{CO}$$

CO $\Rightarrow$ any function $f(x)$ can be written as a linear combination of the $N_k(x)$:

$$f(x) = \int dx'\, f(x')\, \delta(x-x') \qquad \text{CO}$$

$$= \int dx'\, f(x') \sum_k N_k(x) N_k(x')$$

$$= \sum_k N_k(x) \cdot \underbrace{\int dx'\, f(x') N_k(x')}_{\doteq c_k}$$

$$= \sum_k c_k N_k(x)$$

ON $\Rightarrow$ The coefficients c_k are determined uniquely

$$\int dx\, f(x)\, N_k(x) = \int dx \left(\sum_{k'} c_{k'} N_{k'}(x)\right) N_k(x)$$

$$= \sum_{k'} c_{k'} \int dx\, N_{k'}(x)\, N_k(x)$$

$$= \sum_{k'} c_{k'}\, \delta_{kk'} \qquad \text{ON}$$

$$= c_k \quad . \quad \checkmark$$

- Just like expanding any vector in a basis $\vec{v} = \sum_k v_k \vec{e}_k$ basis vectors

ON $\leftrightarrow$ $\vec{e}_k \cdot \vec{e}_{k'} = \delta_{kk'}$

CO $\leftrightarrow$ $\sum_k \vec{e}_k\, \vec{e}_k^{\,T} = I$. identity matrix

Familiar examples:

- If $N_k(x) \leftrightarrow e^{in\phi}$ $n \in \mathbb{Z}$ (integer)

"k" "x"

ON: $\frac{1}{2\pi}\int_0^{2\pi} d\phi \; e^{in\phi} e^{-im\phi} = \delta_{n,m}$

CO: $\sum_n e^{in\phi} e^{-in\phi'} = 2\pi\,\delta(\phi-\phi')$

$$\Rightarrow \begin{cases} f(\phi) = \sum_n c_n e^{in\phi} \\ c_n = \frac{1}{2\pi}\int_0^{2\pi} d\phi \; f(\phi)\, e^{-in\phi} \end{cases}$$

= "Fourier series" for periodic functions

- If $N_k(x) \leftrightarrow e^{ikx}$ $\quad k \in \mathbb{R}$ (reals)

ON $\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \; e^{ikx} e^{-ik'x} = \delta(k-k')$

CO $\quad \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \; e^{ikx} e^{-ikx'} = \delta(x-x')$

$$\Rightarrow \begin{cases} f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \; \tilde{f}(k)\, e^{ikx} \\ \tilde{f}(k) = \int_{-\infty}^{\infty} dx \; f(x)\, e^{-ikx} \end{cases}$$

= "Fourier transform"

- These examples make it clear that the precise set of special functions used depends on the boundaries

e.g. if boundaries are at $x = x_0$ & $x = x_1$ and x_0, x_1 are finite, then get Fourier series; but if x_0 and/or $x_1 = \pm\infty$ then get Fourier transform.

Cartesian coordinates

$$\nabla^2 V = 0 \quad \& \quad V(x,y,z) \doteq X(x)\, Y(y)\, Z(z)$$

$$0 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(X(x)\, Y(y)\, Z(z)\right)$$

$$= (\partial_x^2 X)\, Y Z + X(\partial_y^2 Y) Z + X Y (\partial_z^2 Z)$$

$$0 = \frac{\partial_x^2 X}{X} + \frac{\partial_y^2 Y}{Y} + \frac{\partial_z^2 Z}{Z} \qquad \Leftarrow$$

$$\Rightarrow \quad \frac{\partial_x^2 X}{X} = -k_x^2 \qquad \frac{\partial_y^2 Y}{Y} = -k_y^2 \qquad \frac{\partial_z^2 Z}{Z} = -k_z^2 \qquad *$$

w/ k_x^2, k_y^2, k_z^2 constants &

$$k_x^2 + k_y^2 + k_z^2 = 0. \qquad \cancel{*}\ *$$

$**$ implies not all of k_x^2, k_y^2, k_z^2 can be positive!

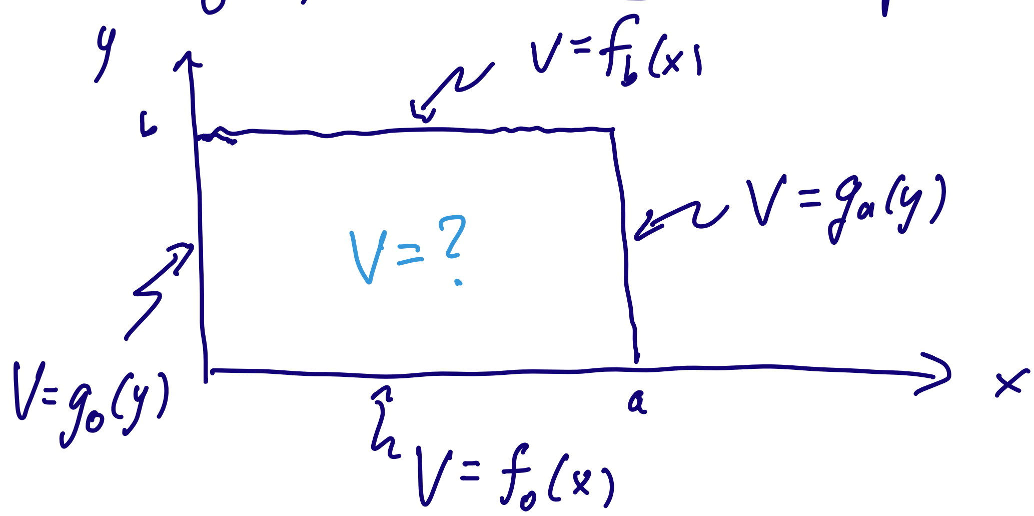
Solutions to $*$: $\partial_x^2 X(x) = -k^2 X(x)$

$$\left.\begin{aligned} \Rightarrow X(x) &= \tilde{A}\cos kx + \tilde{B}\sin kx \\ &= A e^{ikx} + B e^{-ikx} \end{aligned}\right\} \text{ if } k^2 > 0$$

$$\left.\begin{aligned} \Rightarrow X(x) &= \tilde{A}\cosh(\kappa x) + \tilde{B}\sinh(\kappa x) \\ &= A e^{\kappa x} + B e^{-\kappa x} \end{aligned}\right\} \begin{aligned} &\text{if } k^2 = -\kappa^2 < 0 \\ &(\text{so } \kappa^2 > 0) \end{aligned}$$

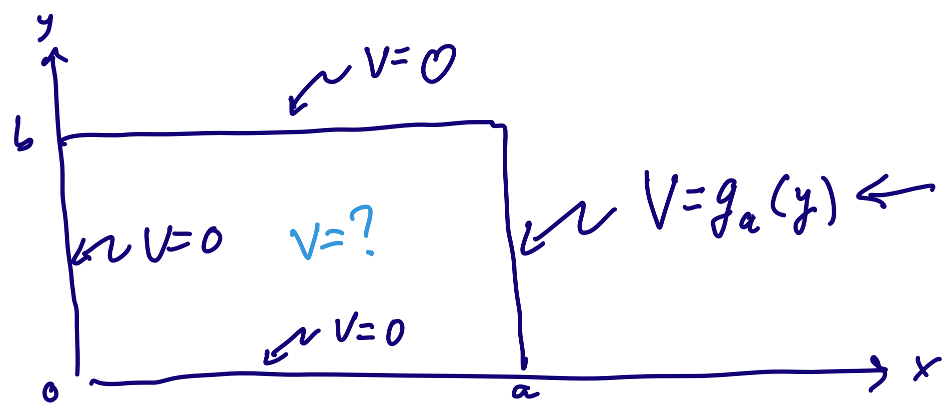
Values k_x^2, k_y^2, k_z^2 can take are determined by the <u>boundary conditions</u>.

E.g., do a 2-d example (i.e., ignore z):



$f_i(x)$, $g_i(y)$ are specified functions

(0) Reduce to four problems like:



$$\nabla^2 V = 0 \;\&\; V = X\cdot Y \Rightarrow \partial_x^2 X = \underbrace{-k^2}_{k_x^2} X \;\&\; \partial_y^2 Y = \underbrace{+k^2}_{-k_y^2} Y$$

(1) Choose $k^2 \geq 0$:

$$\Rightarrow \begin{cases} X_k = A_k \cos(kx) + B_k \sin(kx) & (k \geq 0) \\ Y_k = C_k e^{ky} + D_k e^{-ky} \end{cases}$$

(2) Impose $y=0, b$ boundary conditions:

$V(x,0)=V(x,b)=0 \ \forall x \Rightarrow Y_k(0)=Y_k(b)=0 \quad \forall k \geq 0$

$\Rightarrow \quad C_k + D_k = 0 = C_k e^{kb} + D_k e^{-kb}$

$\Rightarrow \quad D_k = -C_k \ \& \ e^{kb} - e^{-kb} = 0$

$\Rightarrow \quad e^{2kb} = 1 \Rightarrow k = 0 \Rightarrow Y_0 = \text{const} \ ! ?$

(3) So try $\underset{\underset{-k^2}{\parallel\!\!\!\!\parallel}}{\ell^2} \leq 0$: $\begin{cases} X_k = A_k e^{kx} + B_k e^{-kx} \\ Y_k = C_k \cos ky + D_k \sin ky \end{cases}$ $(k \geq 0)$

& $Y_k(0) = Y_k(b) = 0 \ \forall k \geq 0 \Rightarrow$

$$C_k = 0 = C_k \cos kb + D_k \sin kb$$

$$\Rightarrow \sin kb = 0$$

(4) Solve boundary condition to get infinite series of solutions:

$\Rightarrow \quad k = \frac{\pi n}{b}, \ n \in \mathbb{Z}_{\geq 0}$ and $D_k = 1 \leftarrow !$

$$\Rightarrow \begin{cases} X_n = A_n e^{\frac{\pi}{b} n x} + B_n e^{-\frac{\pi}{b} n x} \\ Y_n = \sin\left(\frac{\pi}{b} n y\right) \quad n \geq 1 \leftarrow ! \end{cases}$$

$$\therefore \quad V = \sum_{n=1}^{\infty} \left(A_n e^{\frac{\pi}{b} n x} + B_n e^{-\frac{\pi}{b} n x} \right) \sin\left(\frac{\pi}{b} n y\right)$$

(5) Impose $x = 0, a$ boundary conditions:

$$V(0,y) = 0 \Rightarrow V = \sum_{n=1}^{\infty} (A_n + B_n) \sin\left(\frac{\pi}{b} n y\right) \equiv 0$$

$\Rightarrow$ (completeness) $A_n + B_n = 0 \quad \forall n \geq 1$

$$\therefore \quad \boxed{V = 2 \sum_{n=1}^{\infty} A_n \sinh\left(\frac{\pi}{b} n x\right) \sin\left(\frac{\pi}{b} n y\right)}$$

$$V(a,y) = g_a(y) = 2 \sum_{n=1}^{\infty} A_n \sinh\left(\frac{\pi n a}{b}\right) \sin\left(\frac{\pi}{b} n y\right)$$

(6) Use orthogonality of special functions to compute A_n

$$\int_0^b dy \, \sin\left(\frac{\pi}{b} n y\right) \sin\left(\frac{\pi}{b} m y\right) \overset{?}{\propto} \delta_{n,m}$$

$$= \ldots = \frac{b}{2} \delta_{n,m}$$

$$\left[@\ n=m \quad \int_0^b dy \, \sin^2\left(\frac{\pi}{b} n y\right) = \int_0^b dy \, \frac{1 - \cos\left(\frac{2\pi n}{b} y\right)}{2} = \frac{b}{2} \right]$$

$\therefore$

$$\int_0^b dy\; g_a(y) \sin\left(\frac{\pi}{b} m y\right)$$

$$= 2 \sum_{n=1}^{\infty} A_n \sinh\left(\frac{\pi n a}{b}\right) \int_0^b dy\, \sin\left(\frac{\pi}{b} n y\right) \sin\left(\frac{\pi}{b} m y\right)$$

$$= 2 \sum_{n=1}^{\infty} A_n \sinh\left(\frac{\pi n a}{b}\right) \frac{b}{2} \delta_{n,m}$$

$$= b A_m \sinh\left(\frac{\pi m a}{b}\right)$$

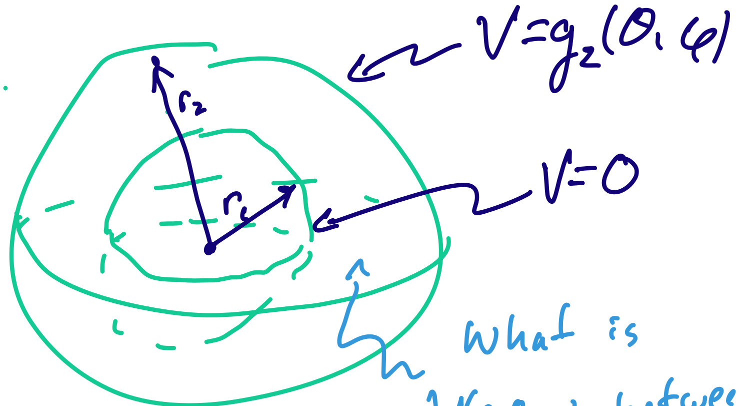
$$\Rightarrow \boxed{A_m = \frac{1}{b \sinh\left(\frac{\pi m a}{b}\right)} \int_0^b dy\; g_a(y) \sin\left(\frac{\pi}{b} m y\right)}$$

Solved!

<u>Example: Spherical coordinates</u>

Region:

$$\left\{ \begin{array}{l} r_1 < r < r_2 \\ \forall\, \theta, \varphi \end{array} \right\}$$



What is $V(r, \theta, \varphi)$ between the 2 spheres?

with boundary conditions:

$$V(r = r_1, \theta, \varphi) = 0$$

$$V(r = r_2, \theta, \varphi) = g_2(\theta, \varphi) \leftarrow \text{specified fnc.}$$

$$V = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

$$0 = \nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial\varphi^2}$$

$\Rightarrow$

$$0 = \partial_r(r^2\partial_r R)\,\Theta\,\Phi + \frac{\partial_\theta(\sin\theta\,\partial_\theta\Theta)}{\sin\theta} R\,\Phi + \frac{R\,\Theta}{\sin^2\theta}\partial_\varphi^2\Phi$$

$\Rightarrow$

$$0 = \underbrace{\frac{\partial_r(r^2\partial_r R)}{R}}_{\doteq\,\lambda} + \frac{\partial_\theta(\sin\theta\,\partial_\theta\Theta)}{\Theta\sin\theta} + \frac{\partial_\varphi^2\Phi}{\sin^2\theta\,\Phi}$$

$\Rightarrow$ $\boxed{\partial_r(r^2\partial_r R) = \lambda R}$ (R) & $\dfrac{\partial_\theta(\sin\theta\,\partial_\theta\Theta)}{\Theta\sin\theta} + \dfrac{\partial_\varphi^2\Phi}{\sin^2\theta\,\Phi} = -\lambda$, $\lambda \in \mathbb{R}$

$\times \sin^2\theta$: $$\frac{\sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta\Theta)}{\Theta} + \lambda\sin^2\theta + \underbrace{\frac{\partial_\varphi^2\Phi}{\Phi}}_{\doteq\,-m^2} = 0$$

$\Rightarrow$ $\boxed{\partial_\varphi^2\Phi = -m^2\Phi}$ (Φ), $m^2 \in \mathbb{R}$

&

$\boxed{\sin\theta\,\partial_\theta(\sin\theta\,\partial_\theta\Theta) + \lambda\sin^2\theta\,\Theta = m^2\Theta}$ (Θ)

- Look at Φ eqn first $\Rightarrow$

$$\Rightarrow \begin{cases} \Phi_m = e^{im\varphi} & m \in \mathbb{R} \quad (\text{assuming } m^2>0) \\ \Phi_\mu = e^{\mu\varphi} & \mu \in \mathbb{R} \quad (\;'' \quad m^2=-\mu^2<0) \end{cases}$$

What are boundary conditions on $\Phi_{m,\mu}$?

Periodicity in φ: $\varphi \sim \varphi + 2\pi \Rightarrow$

$$\Phi_{m,\mu}(\varphi + 2\pi) = \Phi_{m,\mu}(\varphi) .$$

There is no solution (except $\mu = 0$) for Φ_μ. $\times$

For Φ_m get

$$e^{2\pi i m} = 1 \Rightarrow \boxed{m \in \mathbb{Z}} .$$

- Look at Θ eqn now.

$$\sin\theta\, \partial_\theta (\sin\theta\, \partial_\theta \Theta) + \lambda \sin^2\theta\, \Theta = m^2 \Theta$$

What are boundary conditions?

At $\theta = 0, \pi$ want $\Theta(\theta)$ to be regular

$$\Rightarrow \ldots \quad \boxed{\lambda = \ell(\ell+1) \quad \text{with } \ell \in \mathbb{Z} \geq |m|}$$

Solutions of Θ eqn are then the "associated Legendre polynomials" $P_\ell^m(\cos\theta, \sin\theta)$ ...

- For this course we will stick to situations where only $m = 0$ contributes.

This means only $\Phi_0(\varphi) = 1$ is allowed, which means we are restricting to problems where there is no φ-dependence, i.e. there is rotational symmetry around the z-axis.

So, we have to modify our problem to have boundary condition:

no φ-dependence

$$V(r=r_2, \theta, \varphi) = g_2(\theta)$$

- When $m=0$, the solutions of the Θ eqn regular at $\theta = 0, \pi$ are the "Legendre polynomials"

$$\boxed{\begin{aligned} &\Theta_\ell(\theta) = P_\ell(\cos\theta) \qquad \ell \in \mathbb{Z} \geq 0 \\ &\text{w/} \quad P_\ell(x) \doteq \frac{1}{2^\ell \cdot \ell!} \left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell \end{aligned}}$$

(See table 3.1 in Griffiths for list of first few P_ℓ's.)

Properties of $P_\ell(x)$:

$$P_\ell(1) = 1$$

$$P_\ell(-x) = (-)^\ell P_\ell(x)$$

$$\boxed{\begin{aligned} \int_{-1}^{1} dx\, P_m(x)\, P_n(x) &= \delta_{m,n} \cdot \frac{2}{2n+1} \\ \| \\ \int_0^\pi \sin\theta\, d\theta\, P_m(\cos\theta)\, P_n(\cos\theta) \end{aligned}}$$

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(x)\, P_\ell(x') = \delta(x-x')$$

- Now look at R equ:

$$\partial_r(r^2 \partial_r R) = \ell(\ell+1) R$$

Notice that is invariant under rescalings $r \to \alpha r$, so given solutions

$$R \sim r^a$$

Plug in $\Rightarrow \cdots \quad a(a+1) = \ell(\ell+1)$

$\Rightarrow \quad a = \ell \quad$ or $\quad a = -\ell - 1$

So general R_ℓ solution is

$$\boxed{R_\ell(r) = A_\ell r^\ell + B_\ell r^{-\ell-1}}$$

- Putting this all together, we have

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)}$$

only good if (1) no φ-dependence, & (2) whole $0 \le \theta \le \pi$ range!

- Apply $r = r_1$ & $r = r_2$ boundary conditions:

$$V(r_1, \theta) = 0 = \sum_{\ell=0}^{\infty} \left(A_\ell r_1^\ell + B_\ell r_1^{-\ell-1}\right) P_\ell(\cos\theta)$$

Completeness of P_ℓ's $\Rightarrow$

(1) $$\boxed{A_\ell r_1^{\ell} + B_\ell r_1^{-\ell-1} = 0} \quad \forall \ell \geq 0$$

$$V(r_2,\theta) = g_2(\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell r_2^{\ell} + B_\ell r_2^{-\ell-1}\right) P_\ell(\cos\theta)$$

Use orthogonality of P_ℓ's :

$$\boxed{\int_0^{\pi} \sin\theta \, d\theta \; g_2(\theta) \, P_n(\cos\theta)}$$

$$= \sum_{\ell=0}^{\infty} \left(A_\ell r_2^{\ell} + B_\ell r_2^{-\ell-1}\right) \cdot$$

$$\cdot \int_0^{\pi} \sin\theta \, d\theta \; P_\ell(\cos\theta) \, P_n(\cos\theta)$$

$$= \sum_{\ell=0}^{\infty} \left(A_\ell r_2^{\ell} + B_\ell r_2^{-\ell-1}\right) \cdot \frac{2\delta_{\ell,n}}{2n+1}$$

$$\boxed{= \frac{2}{2n+1}\left(A_n r_2^{n} + B_n r_2^{-n-1}\right)} \quad (2)$$

(1) & (2) $\Rightarrow$ determine A_ℓ, B_ℓ $\forall \ell$. ✓