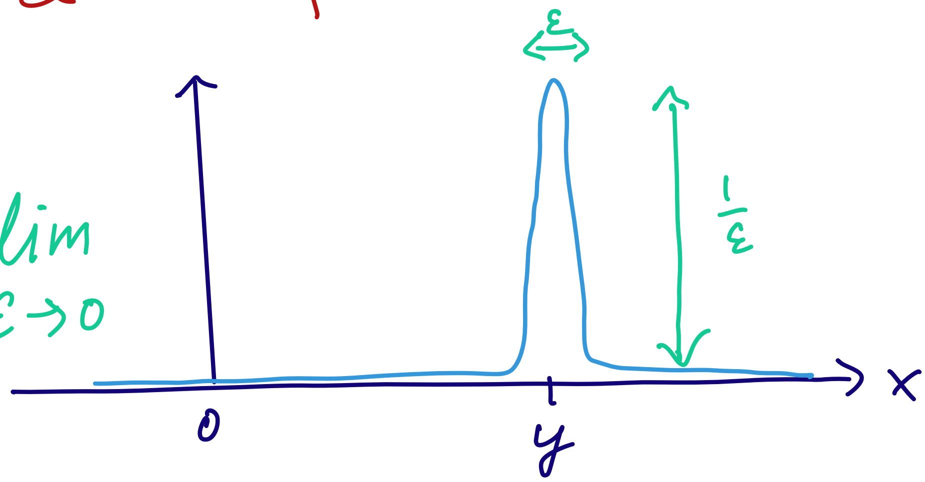
LECTURE 3

1.5 Dirac delta function

- 1-d

$$\delta(x-y) = \lim_{\varepsilon \to 0}$$



"infinitely thin bump" at $x=a$ of area 1

$$\Rightarrow \int_a^b dx\, \delta(x-y) = \begin{cases} 1 & \text{if } a<y<b \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow f(x)\,\delta(x-y) = f(y)\,\delta(x-y)$$
for any continuous $f(x)$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} dx\, f(x)\,\delta(x-y) = f(y) \quad \underline{\text{for all } f}} \quad (*)$$

- Key (defining) property of "$\delta(x-y)$"
- δ-functions always appear in integrals in physical expressions.

- Turn (*) around to define "generalized functions" or "distributions":

2 distributions $D_1(x)$ & $D_2(x)$ are equal iff

$$\int_{-\infty}^{\infty} dx\, f(x)\, D_1(x) = \int_{-\infty}^{\infty} dx\, f(x)\, D_2(x)$$

for all $f(x)$.

- Example: $\boxed{\delta(ax-b) = \frac{1}{|a|}\,\delta\!\left(x-\frac{b}{a}\right)}$ $(a\neq 0)$

Proof:

$$I_1 \doteq \int_{-\infty}^{\infty} dx\, f(x)\, \delta(ax-b)$$

$$I_2 \doteq \int_{-\infty}^{\infty} dx\, f(x)\, \frac{1}{|a|}\,\delta\!\left(x-\frac{b}{a}\right)$$

Change variables $\bar{x} \doteq ax-b$ in I_1:

$$\Rightarrow \quad d\bar{x} = a\, dx$$

$$I_1 = \int_{-a\cdot\infty}^{+a\cdot\infty} \frac{1}{a}\, d\bar{x}\, f\!\left(\frac{\bar{x}+b}{a}\right) \delta(\bar{x})$$

$$= \begin{cases} \int_{-\infty}^{\infty} \frac{1}{a} d\bar{x}\, f\left(\frac{\bar{x}+b}{a}\right)\delta(\bar{x}) & \text{if } a>0 \\ \int_{\infty}^{-\infty} \frac{1}{a} d\bar{x}\, f\left(\frac{\bar{x}+b}{a}\right)\delta(\bar{x}) & \text{if } a<0 \end{cases}$$

$$= \underbrace{\left(\int_{-\infty}^{\infty} d\bar{x}\, f\left(\frac{\bar{x}+b}{a}\right)\delta(\bar{x})\right)}_{f\left(\frac{b}{a}\right)}^{\text{by } (*)} \cdot \underbrace{\begin{cases} \frac{1}{a} & \text{if } a>0 \\ -\frac{1}{a} & \text{if } a<0 \end{cases}}_{\frac{1}{|a|}}$$

$$\therefore\ I_1 = f\left(\frac{b}{a}\right) \cdot \frac{1}{|a|} .$$

$$I_2 = \frac{1}{|a|} \int_{-\infty}^{\infty} dx\, f(x)\, \delta\left(x - \frac{b}{a}\right)$$

$$= \frac{1}{|a|} \cdot f\left(\frac{b}{a}\right) \qquad \text{by } (*) .$$

$=$ ✓

• <u>3-d</u> In Cartesian coords:

$$\delta^3(\vec{r} - \vec{r}\,') \doteq \delta(x-x')\,\delta(y-y')\,\delta(z-z')$$

$$\Rightarrow \boxed{\int_{\text{all space}} d\tau\, \delta^3(\vec{r}-\vec{r}\,')\, f(\vec{r}) = f(\vec{r}\,') \qquad \forall f} \quad (*)_{3d}$$

just because $\int_{\text{all space}} d\tau = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz$.

Example: $$\boxed{\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{\mathscr{r}})}$$

(Recall: $\vec{\mathscr{r}} \doteq \vec{r} - \vec{r}\,'$, and $\vec{\nabla} \doteq \vec{\nabla}_{\vec{r}} = \vec{\nabla}_{\vec{\mathscr{r}}}$.)

Proof: Want to show for all $f(\vec{r})$
$I_1 = I_2$ where

$$I_1 \doteq \int d\tau\, f(\vec{r})\, \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right),$$

$$I_2 \doteq \int d\tau\, f(\vec{r})\, 4\pi\,\delta^3(\vec{\mathscr{r}}).$$

First, $I_2 = 4\pi \int d\tau\, f(\vec{r})\, \delta^3(\vec{r}-\vec{r}\,') = 4\pi f(\vec{r}\,')$
by $\circledast_{3d}$.

Next $$\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \vec{\nabla}_{\vec{\mathscr{r}}}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$= \frac{1}{\mathscr{r}^2}\frac{\partial}{\partial \mathscr{r}}\left(\mathscr{r}^2 \frac{1}{\mathscr{r}^2}\right) = 0, \quad \underline{\mathscr{r} > 0}$$

where we have gone to spherical coordinates centered on $\vec{r}\,'$, so that

$\mathscr{r}$ is the radial variable.

Therefore, for any continuous $f(\vec{r})$, since the integrand of I_1 is 0 except at $\vec{r} = \vec{r}\,'$, we have

$$I_1 = \int d\tau\, f(\vec{r})\, \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \int d\tau\, f(\vec{r}\,')\, \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

$$= f(\vec{r}\,') \int d\tau\, \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

all space

$$= f(\vec{r}\,') \int_{\mathscr{r}<R} d\tau\, \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

ball of radius R centered on $\vec{r}\,'$

Now use divergence theorem to evaluate

$$\int_{\mathscr{r}<R} d\tau\, \vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \oint_{\mathscr{r}=R} d\vec{a}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)$$

sphere of radius R centered on $\vec{r}\,'$

$$= \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi\, R^2\, \hat{\mathscr{r}}\cdot\left(\frac{\hat{\mathscr{r}}}{R^2}\right)$$

$$= 4\pi$$

where I used spherical coordinates centered at $\vec{r}'$, so r = radial variable.

$$\therefore \quad I_1 = f(\vec{r}')\cdot 4\pi \ . \checkmark$$

- Since $\vec{\nabla}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$ (check!)

$$\Rightarrow \quad \boxed{\nabla^2 \frac{1}{r} = -4\pi\,\delta^3(\vec{r})}$$

1.6 Curl-less & divergence-less fields

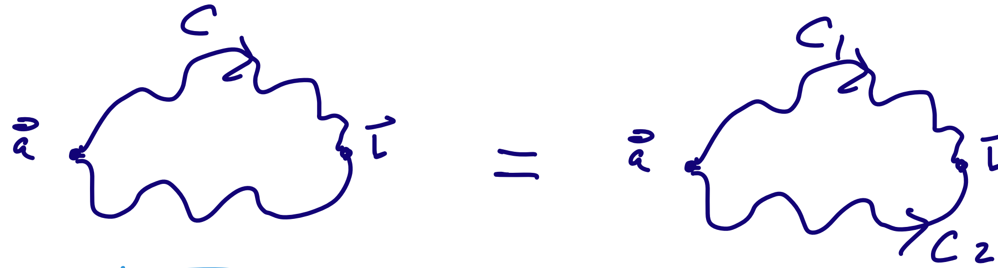
- Curl-less fields: $\boxed{\vec{\nabla}\times\vec{A} = 0 \text{ everywhere.}}$

Then for any surface S

$$0 = \int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) = \oint_{\partial S} d\vec{l}\cdot\vec{A}$$

So for any closed loop $C = \partial S$:

- $$0 = \oint_C d\vec{l}\cdot\vec{A} = \int_{C_1} d\vec{l}\cdot\vec{A} - \int_{C_2} d\vec{l}\cdot\vec{A}$$



So $\boxed{\displaystyle\int_{\vec{a}}^{\vec{b}} d\vec{l}\cdot\vec{A} \text{ is independent of path.}}$

- There is a scalar field f such that

$$\boxed{\vec{A} = \vec{\nabla} f}$$

(Harder to prove; only true if $\vec{\nabla}\times\vec{A} = 0$ everywhere in 3-d space.)

– Note f is not unique because if

$$f' = f + c \quad \text{w/} \quad c = \text{constant},$$

then $\vec{\nabla} f' = \vec{\nabla} f$.

- Divergence-less fields $\boxed{\vec{\nabla}\cdot\vec{A} = 0 \text{ everywhere}}$

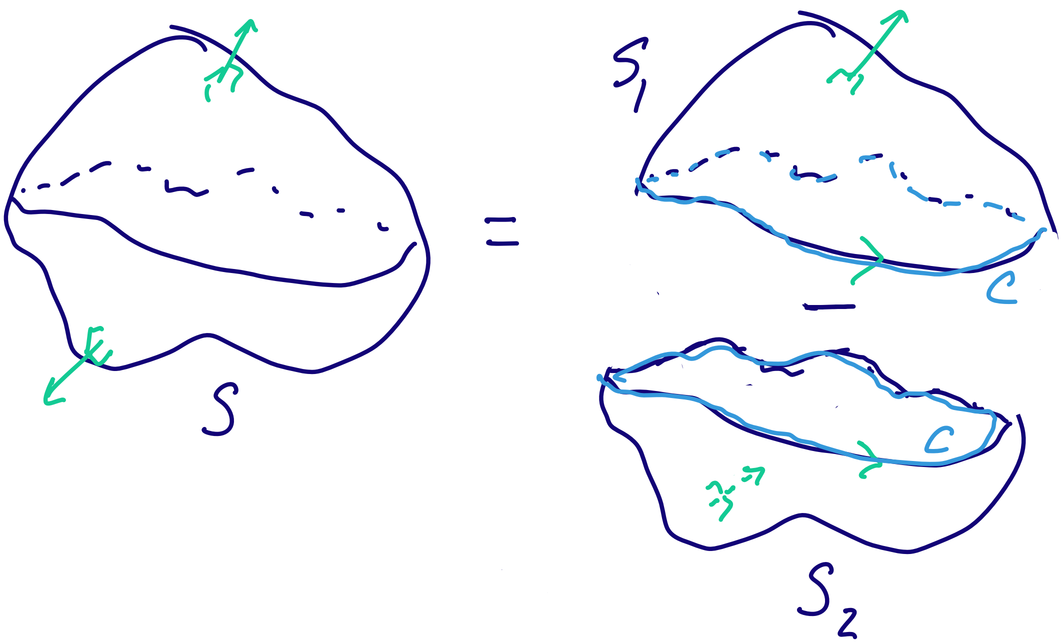
Then for any region V

$$0 = \int_V d\tau\, \vec{\nabla}\cdot\vec{A} = \oint_{\partial V} d\vec{a}\cdot\vec{A}$$

So for any closed surface S

$$\boxed{0 = \oint_S d\vec{a}\cdot\vec{A}}$$

$$= \int_{S_1} d\vec{a}\cdot\vec{A} - \int_{S_2} d\vec{a}\cdot\vec{A}$$



$\therefore \quad \int_{S_1} d\vec{a}\cdot\vec{A} = \int_{S_2} d\vec{a}\cdot\vec{A}$

for any S_1 & S_2 as long as $\partial S_1 = \partial S_2$.

Finally, there exists a vector field $\vec{B}$ such that

$$\boxed{\vec{A} = \vec{\nabla} \times \vec{B}}$$

(Harder to prove; only true if $\vec{\nabla} \cdot \vec{A} = 0$ everywhere in 3-d space.)

- <u>Note</u> $\vec{B}$ is not unique because if

$\vec{B}' = \vec{B} + \vec{\nabla} f$ then $\vec{\nabla} \times \vec{B}' = \vec{\nabla} \times \vec{B}$.