LECTURE 2

1.3 Integral calculus

- Fundamental theorem of calculus

$$\boxed{\int_{\text{region}} \text{derivative}(f) = \int_{\substack{\text{boundary} \\ \text{of region}}} f}$$

"integration is inverse of differentiation"

In 3-d get many versions:

- Scalar:

$$\int_C d\vec{l}\cdot\vec{\nabla} f = \int_{\partial C} f$$

$$\int_S d\vec{a}\cdot(\vec{\nabla}\times\vec{A}) = \int_{\partial S} d\vec{l}\cdot\vec{A}$$

$$\int_V d\tau\ \vec{\nabla}\cdot\vec{A} = \int_{\partial V} d\vec{a}\cdot\vec{A}$$

- Vector:

$$\int_S d\vec{a}\times\vec{\nabla} f = \int_{\partial S} d\vec{l}\ f$$

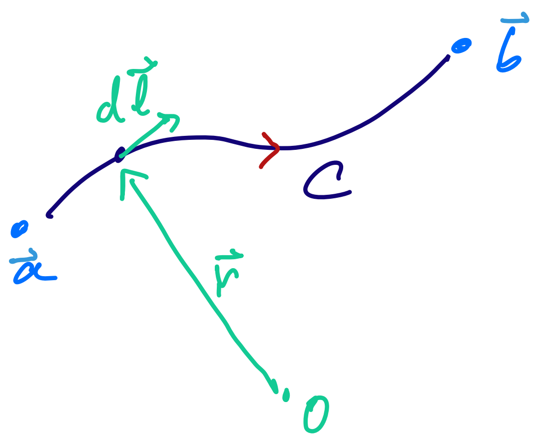
$$\int_V d\tau\ \vec{\nabla} f = \int_{\partial V} d\vec{a}\ f$$

$$\int_V d\tau\,(\vec{\nabla}\times\vec{A}) = \int_{\partial V} d\vec{a}\times\vec{A}$$

- To compute, need to know def'ns:
  - C, S, V — oriented curve, surface, volume
  - $\partial C, \partial S, \partial V$ — boundaries of "
  - $\int_{\partial C}$ — 0-dim'l "integral"
  - $\int_C d\vec{l}$ — 1-dim'l "line integral"
  - $\int_S d\vec{a}$ — 2-dim'l "surface integral"
  - $\int_V d\tau$ — 3-dim'l "volume integral"

- Geometric definitions (coordinate-indep't)

<u>$C \doteq$ oriented curve:</u>



$>$ = orientation

Describe $C = \{ \vec{r}(s) ,\ 0 \le s \le 1 \}$

$\vec{r}(0) = \vec{a}$ & $\vec{r}(1) = \vec{b}$

$d\vec{l}(s)$ = "infinitesimal tangent vector at s" = $\frac{d\vec{r}}{ds}(s)\, ds$

$\int_C d\vec{l}$ = sum of infinitesimal tangent vectors

$= \int_0^1 ds\, \frac{d\vec{r}(s)}{ds}$

$\partial C \doteq$ boundary of curve

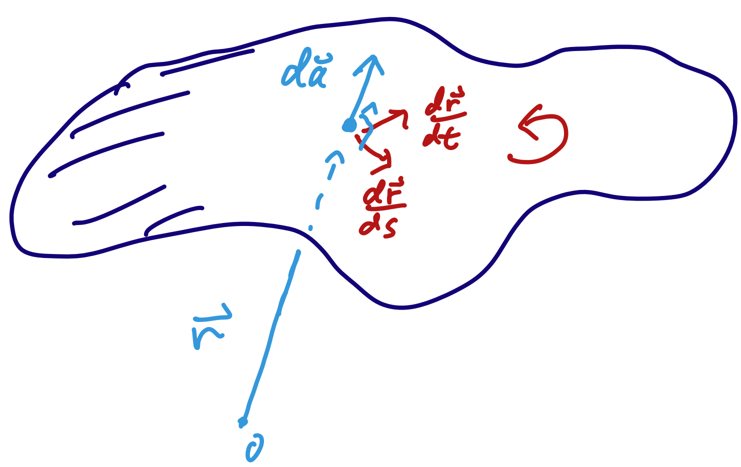
$\doteq \{\vec{b}\} - \{\vec{a}\}$ "difference of endpoints"

$$\int_{\partial C} f \doteq f(\vec{b}) - f(\vec{a}) \doteq f\Big|_{\vec{a}}^{\vec{b}}$$

Note: If C is closed curve $\Rightarrow \vec{b} = \vec{a}$

& $\partial C = \emptyset$ (empty set).

$S \doteq$ oriented surface



$\circlearrowleft$ = orientation

$d\vec{a}$ sign by r.h.r.

Describe $S = \{\vec{r}(s,t),\ 0 \le s \le 1, 0 \le t \le 1\}$

$$\vec{r}(s,t) = x(s,t)\,\hat{x} + y(s,t)\,\hat{y} + z(s,t)\,\hat{z}$$

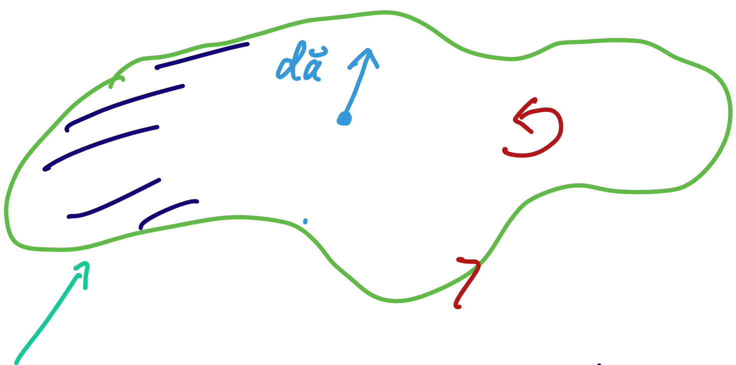
$d\vec{a}$ = infinitesimal normal surface area

$$= ds\,dt\,\frac{d\vec{r}}{ds}\times\frac{d\vec{r}}{dt}$$

$$\int_S d\vec{a} = \int_0^1 ds \int_0^1 dt\, \frac{d\vec{r}}{ds}\times\frac{d\vec{r}}{dt}$$

∂S ≐ boundary of surface = closed curve

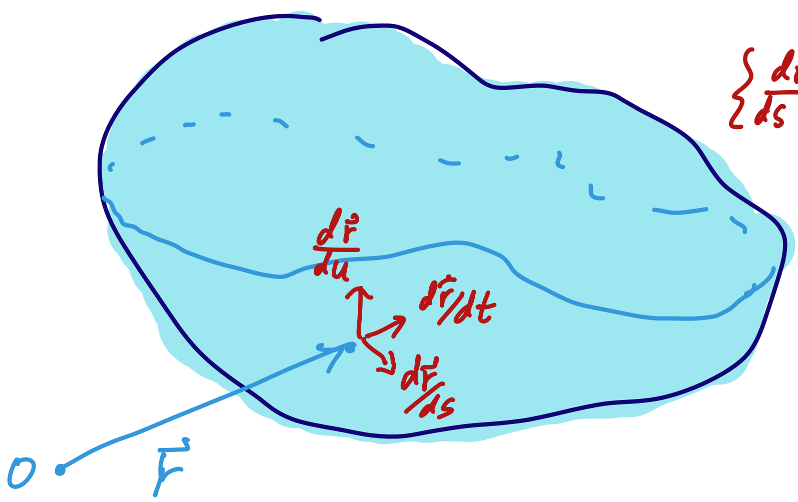
$$\left(\int_{\partial S} d\vec{l} \doteq \oint_{\partial S} d\vec{l}\right)$$



∂S = closed curve w/ orientation induced from that of S.

V ≐ oriented volume

$$\left\{\frac{d\vec{r}}{ds}, \frac{d\vec{r}}{dt}, \frac{d\vec{r}}{du}\right\} \doteq \text{orientation}$$



Describe $V = \{ \vec{r}(s,t,u)\ ,\ 0 \le s,t,u \le 1 \}$.

(e.g., just take $(s,t,u) = (x,y,z)$ cartesian coords)

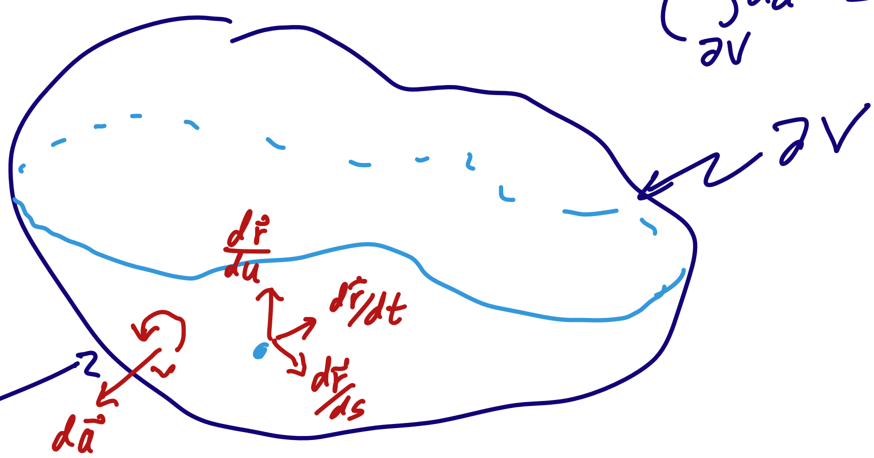
$d\tau \doteq$ (infinitesimal) "volume element"

$$\doteq ds\,dt\,du\ \frac{d\vec{r}}{ds} \cdot \left(\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{du} \right)$$

$$\int_V d\tau = \int_0^1 ds \int_0^1 dt \int_0^1 du\ \frac{d\vec{r}}{ds} \cdot \left(\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{du} \right)$$

<u>$\partial V \doteq$ oriented boundary of volume = closed surface</u>

$$\left(\int_{\partial V} d\vec{a} \doteq \oint_{\partial V} d\vec{a} \right)$$



∂V = closed surface with orientation induced from that of V:

$d\vec{a}$ points <u>out of</u> V if $\frac{d\vec{r}}{ds} \cdot \left(\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{du} \right) > 0$ "rhr"

" " <u>into</u> V " " < 0. "lhr"

• Analytic definitions in Cartesian coordinates

$$C = \{ \vec{r}(s) = x(s)\hat{x} + y(s)\hat{y} + z(s)\hat{z} \,,\quad 0 \le s \le 1 \}$$

$$\Rightarrow \int_C d\vec{l} = \int_0^1 ds \, \frac{d\vec{r}}{ds} = \int_0^1 ds \left(\frac{dx}{ds}\hat{x} + \frac{dy}{ds}\hat{y} + \frac{dz}{ds}\hat{z} \right)$$

$$= \text{"} \int_C (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}) \text{"}$$

$$S = \{ \vec{r}(s,t) = x(s,t)\hat{x} + y(s,t)\hat{y} + z(s,t)\hat{z} \,,\quad 0 \le s,t \le 1 \}$$

$$\Rightarrow \int_S d\vec{a} = \iint ds\,dt \, \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \iint ds\,dt \, \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_s x & \partial_s y & \partial_s z \\ \partial_t x & \partial_t y & \partial_t z \end{pmatrix}$$

$$V = \{ \vec{r}(s,t,u) = x(s,t,u)\hat{x} + y(s,t,u)\hat{y} + z(s,t,u)\hat{z} \,,\quad 0 \le s,t,u \le 1 \}$$

$$\Rightarrow \int_V d\tau = \iiint ds\,dt\,du \, \frac{\partial \vec{r}}{\partial s} \cdot \left(\frac{\partial \vec{r}}{\partial t} \times \frac{\partial \vec{r}}{\partial u} \right) = \iiint ds\,dt\,du \cdot \det \begin{pmatrix} \partial_s x & \partial_s y & \partial_s z \\ \partial_t x & \partial_t y & \partial_t z \\ \partial_u x & \partial_u y & \partial_u z \end{pmatrix}$$

• Implicit description of surfaces

Can also describe surface as solution of an equation:

real function

$$S = \{ \vec{r} \mid u(\vec{r}) = 0 \}$$

Find (any) 2 functions $s(\vec{r}), t(\vec{r})$, such that

$$0 < \vec{\nabla} s \cdot (\vec{\nabla} t \times \vec{\nabla} u) = \underbrace{\det \begin{pmatrix} \partial_x s & \partial_y s & \partial_z s \\ \partial_x t & \partial_y t & \partial_z t \\ \partial_x u & \partial_y u & \partial_z u \end{pmatrix}}_{\doteq J}$$

Then (s,t) parameterize S, $d\vec{a} \propto \vec{\nabla} u$, and

$$\int_S d\vec{a} = \int ds\,dt \; J^{-1} \vec{\nabla} u \Big|_{u=0}$$

- Integration by parts: integrate Leibniz rule and use fundamental theorem.

$$d(fg) = (df)\,g + f\,(dg) \Rightarrow \left(\begin{matrix}\text{Leibniz} \\ \text{rule}\end{matrix}\right)$$

$$\int_R g\,df + \int_R f\,dg = \int_R d(fg) \underset{\uparrow}{=} \int_{\partial R} fg$$

(fundamental theorem of calculus)

1 variable:

$$\int_a^b dx \left(\frac{df}{dx} g + \frac{dg}{dx} f \right) = fg \Big|_a^b$$

or,

$$\int_a^b dx \cdot f \cdot \frac{dg}{dx} = -\int_a^b dx \cdot \frac{df}{dx} \cdot g + fg \Big|_a^b$$

3-dimensions: Just apply same logic to any 3-d Leibniz rule & fund. thm.

<u>E.g.</u> $\vec{\nabla} \times (f\vec{A}) = \vec{\nabla} f \times \vec{A} + f\, \vec{\nabla} \times \vec{A}$

$$\Rightarrow \int_S d\vec{a} \cdot \vec{\nabla} \times (f\vec{A}) = \int_S d\vec{a} \cdot (\vec{\nabla} f \times \vec{A}) + \int_S d\vec{a} \cdot (\vec{\nabla} \times \vec{A}) f$$

$$\parallel$$

$$\int_{\partial S} d\vec{l} \cdot \vec{A} f \qquad \ldots$$

1.4 Curvilinear coordinates

- Cartesian coordinates are (x, y, z) and we write $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

- We can use a different set of coordinates $(u, v, w) = (u(\vec{r}), v(\vec{r}), w(\vec{r}))$ for some functions (u, v, w)

- Define at each point $\vec{r}$ a basis of unit vectors pointing in the direction of increase of u, v, or w (keeping the other two coordinates fixed):

$$\hat{u} \doteq \frac{\partial_u \vec{r}}{|\partial_u \vec{r}|}, \quad \hat{v} \doteq \frac{\partial_v \vec{r}}{|\partial_v \vec{r}|}, \quad \hat{w} \doteq \frac{\partial_w \vec{r}}{|\partial_w \vec{r}|}.$$

Note that this basis of unit vectors varies w/ $\vec{r}$: $\hat{u} = \hat{u}(\vec{r})$, etc.

- If you are clever, you can also pick (u, v, w) such that these unit vectors are mutually orthogonal,

$$\hat{u} \cdot \hat{v} = \hat{v} \cdot \hat{w} = \hat{w} \cdot \hat{u} = 0,$$

and have the same orientation as $\hat{x}, \hat{y}, \hat{z}$:

$$\hat{u} \times \hat{v} = \hat{w}, \quad \hat{v} \times \hat{w} = \hat{u}, \quad \hat{w} \times \hat{u} = \hat{v}.$$

- Any vector field $\vec{A}$ can be expanded in this basis:

$$\vec{A}(\vec{r}) = A_u(\vec{r})\, \hat{u} + A_v(\vec{r})\, \hat{v} + A_w(\vec{r})\, \hat{w}.$$

In particular,

$$d\vec{r} = (\partial_u \vec{r})\,du + (\partial_v \vec{r})\,dv + (\partial_w \vec{r})\,dw$$

$$= |\partial_u \vec{r}|\,du\,\hat{u} + |\partial_v \vec{r}|\,dv\,\hat{v} + |\partial_w \vec{r}|\,dw\,\hat{w}$$

Griffiths defines in Appendix A:

$$\left(f \doteq |\partial_u \vec{r}| \,,\, g \doteq |\partial_v \vec{r}| ,\, h \doteq |\partial_w \vec{r}| \right)$$

- Then using the chain rule to go from the (x,y,z) coordinates to the (u,v,w) coords, you find (... lots of calculation ...)

$$\vec{\nabla}\varphi = \hat{u}\,\frac{1}{f}\partial_u\varphi + \hat{v}\,\frac{1}{g}\partial_v\varphi + \hat{w}\,\frac{1}{h}\partial_w\varphi$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{fgh}\left[\partial_u(ghA_u) + \partial_v(fhA_v) + \partial_w(fgA_w)\right]$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{gh}\left[\partial_v(hA_w) - \partial_w(gA_v)\right]\hat{u}$$
$$+ \frac{1}{hf}\left[\partial_w(fA_u) - \partial_u(hA_w)\right]\hat{v}$$
$$+ \frac{1}{fg}\left[\partial_u(gA_v) - \partial_v(fA_u)\right]\hat{w}$$

$$\nabla^2 \varphi = \frac{1}{fgh}\left[\partial_u\left(\frac{gh}{f}\partial_u\varphi\right) + \partial_v\left(\frac{fh}{g}\partial_v\varphi\right) + \partial_w\left(\frac{fg}{h}\partial_w\varphi\right)\right]$$

$$d\vec{l} = f\,du\,\hat{u} + g\,dv\,\hat{v} + h\,dw\,\hat{w}$$

$$d\vec{a}\Big|_u = \pm gh\,dv\,dw\,\hat{u}$$

$$d\vec{a}\Big|_v = \pm fh\,du\,dw\,\hat{v}$$

$$d\vec{a}\Big|_w = \pm fg\,du\,dv\,\hat{w}$$

$$d\tau = fgh\;du\,dv\,dw$$

- While there are ∞'ly many curvilinear coordinate systems, we will use only 2:

	u	v	w	f	g	h
spherical	r	θ	ϕ	1	r	$r\sin\theta$
cylindrical	s	ϕ	z	1	s	1

- Spherical coords:

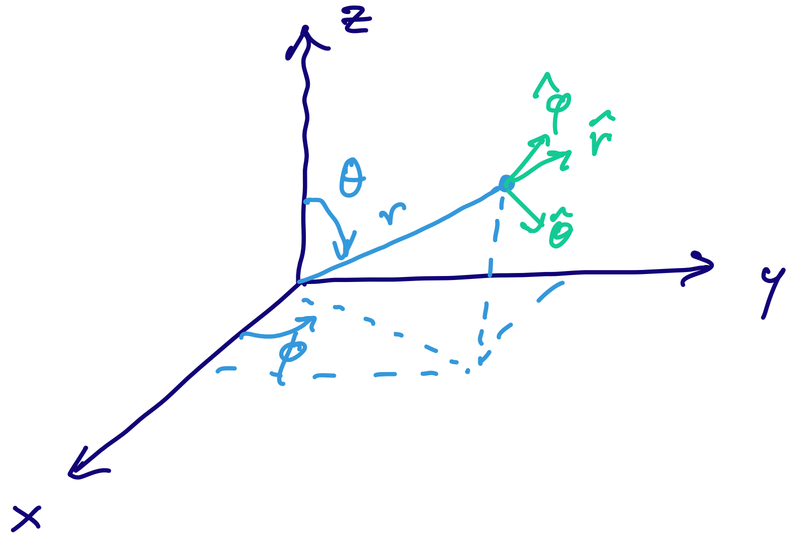
$x = r\sin\theta\cos\phi$ $\qquad\qquad r = \sqrt{x^2+y^2+z^2}$

$y = r\sin\theta\sin\phi$ $\qquad\qquad \phi = \tan^{-1}(y/x)$ "azimuthal angle"

$z = r\cos\theta$ $\qquad\qquad$ $\theta = \tan^{-1}\left(\dfrac{\sqrt{x^2+y^2}}{z}\right)$ "polar angle"

$0 \le r < \infty$ $\qquad$ $0 \le \theta \le \pi$ $\qquad$ $0 \le \phi < 2\pi$ & $\phi \sim \phi + 2\pi$

periodic!



Coordinate singularities at:

$r = 0$ $\quad$ (θ, ϕ ill-defined)

$r \neq 0$ $\quad$ $\theta = 0$ or π $\quad$ (ϕ ill-defined)

$\vec{r} = r\,\hat{r}$, $\quad d\vec{\ell} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$, $\quad d\tau = r^2 \sin\theta\, dr\, d\theta\, d\phi$

- <u>Cylindrical coords</u>:

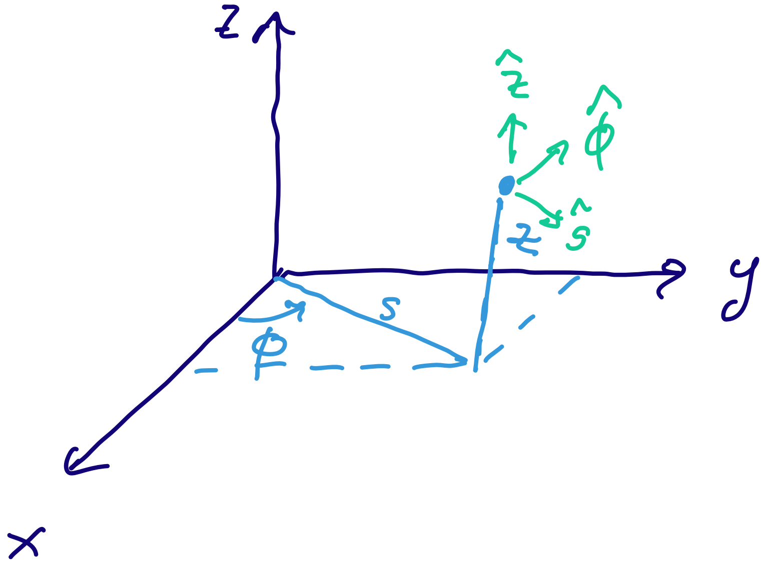
$x = s\cos\phi$ $\qquad\qquad$ $s = \sqrt{x^2+y^2}$

$y = s\sin\phi$ $\qquad\qquad$ $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

$z = z$ $\qquad\qquad$ $z = z$

$0 \le s < \infty$, $\quad$ $0 \le \phi < 2\pi$ & $\phi \sim \phi + 2\pi$, $\quad -\infty < z < \infty$

periodic!



$$\vec{r} = s\hat{s} + z\hat{z}, \quad d\vec{l} = ds\,\hat{s} + s\,d\phi\,\hat{\phi} + dz\,\hat{z}, \quad d\tau = s\,ds\,d\phi\,dz$$