LECTURE 1

Physics 3020

Intro to Electricity & Magnetism

All course info at webpage

homepages.uc.edu/~argyrepc

Text: D. Griffiths "Introduction to electrodynamics" (3rd ed, 1999)

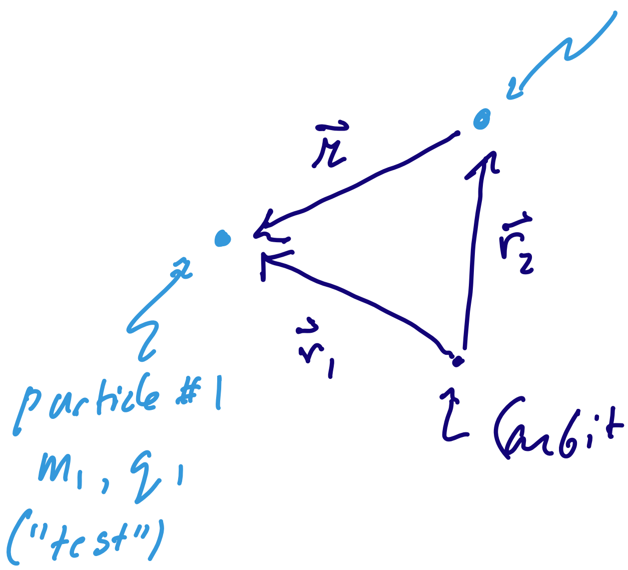
Fall semester outline: "Non-relativistic E&M"

0. Overview
1. Vector calculus review
2. Electrostatics in vacuum
3. Math for linear elliptic P.D.E.s
4. Electrostatics in matter
5. Magnetostatics in vacuum
6. Magnetostatics in matter

0. Overview

- Electricity:

particle #2 : mass m_2 (kg)
("source") charge q_2 (C)



$$\vec{r} \doteq \vec{r}_1 - \vec{r}_2$$

$$r \doteq |\vec{r}|$$

$$\hat{r} \doteq \vec{r}/|\vec{r}|$$

(arbitrary origin of coordinates)

$$m_1 \vec{a}_1 \doteq m_1 \frac{d^2 \vec{r}_1}{dt^2} = \frac{q_1 q_2}{4\pi\epsilon_0} \cdot \frac{\hat{r}}{r^2} \doteq \vec{F}_1^{\,\text{Coulomb}}$$

$\boxed{q_i \text{: positive or negative}}$

experimentally determined constant:

$$\epsilon_0 \simeq 8.85 \times 10^{-12} \cdot \frac{C^2}{N\,m^2}$$

Write $\vec{F}_1^{\,\text{Coul}} \doteq q_1 \underbrace{\vec{E}(\vec{r}_1)}_{'''}$

"Electric field at $\vec{r}_1$ due to q_2"

Then can rewrite Coulomb force law:

$$\vec{F}^{Coul}(\vec{r}) = \underset{\uparrow \text{ test charge}}{q}\,\vec{E}(\vec{r}) \quad \& \quad \begin{cases} \vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{\epsilon_0}\rho(\vec{r}) \\ \vec{\nabla}\times\vec{E}(\vec{r}) = 0 \end{cases}$$

where $\rho(\vec{r}) \doteq$ density of source charges at $\vec{r}$.
(C/m^3)

• <u>Gravity</u>:

$$m_1\vec{a}_1 \doteq m_1\frac{d^2\vec{r}_1}{dt^2} = -\underset{\uparrow}{G}m_1m_2\cdot\frac{\hat{r}}{r^2} \doteq \vec{F}_1^{Newton}$$

experimentally determined constant:

$\boxed{m_i > 0}$.

$$G \cong 6.67\times10^{-11}\cdot\frac{N\,m^2}{kg^2}$$

Write $\vec{F}_1^{Newton} \doteq m_1\underbrace{\vec{g}(\vec{r}_1)}_{\shortparallel}$

"Gravitational field at $\vec{r}_1$ due to m_2"

$$\vec{F}^{Newton}(\vec{r}) = \underset{\uparrow \text{ test mass}}{m}\,\vec{g}(\vec{r}) \quad \& \quad \begin{cases} \vec{\nabla}\cdot\vec{g}(\vec{r}) = -4\pi G\cdot\rho(\vec{r}) \\ \vec{\nabla}\times\vec{g}(\vec{r}) = 0 \end{cases}$$

where $\rho(\vec{r}) \doteq$ density of source masses at $\vec{r}$.
(kg/m^3)

- Similar forms, but electric charge ~ "C" = "Coulomb", new dimensionful quantity (in addition to mass ~ kg, length ~ m, time ~ s).

  In both cases fields ($\vec{E}, \vec{g}$) are just notational/calculational conveniences.

- <u>Electrodynamics</u> & <u>General relativity</u>: more precise theories of electricity & gravity for motion of particles including corrections of order v/c & smaller.

$$v \doteq \left|\frac{d\vec{r}}{dt}\right| = \text{velocity of charged particle}$$
$$c \doteq 3.00 \times 10^8 \frac{m}{s} = \text{speed of light}$$

E.g. Electrodynamics find:

$$\underbrace{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}_{\text{Lorentz force law}}$$

new "magnetic" force field
new constant
charge current density ($C/m^3 s$)

$$\underbrace{\begin{aligned} \vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0}\rho \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}}_{\text{Maxwell's equations}}$$

... Realize: $\vec{E}(\vec{r},t)$ & $\vec{B}(\vec{r},t)$ carry energy at speed $\frac{1}{\sqrt{\epsilon_0\mu_0}} \doteq c$ speed of light

By changing relative units of $q, \vec{E}, \vec{B}$ can rewrite electrodynamics as

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho$$
$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0$$
$$\vec{\nabla}\times\vec{B} = \frac{1}{c}\left(4\pi\vec{J} + \frac{\partial\vec{E}}{\partial t}\right)$$

Makes clear $c\to\infty \Rightarrow \vec{B}\to 0$ ...

Define:

$$x^\mu \doteq (x^0, x^1, x^2, x^3) = (ct, x, y, z)$$

$$\partial_\mu \doteq \frac{\partial}{\partial x^\mu}$$

$$F^{\mu\nu} \doteq \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ - & 0 & B_z & -B_y \\ - & - & 0 & B_x \\ - & - & - & 0 \end{pmatrix}$$

"EM field strength tensor"

$$J^\mu \doteq (c\rho, J_x, J_y, J_z)$$

"4-current density"

Then Maxwell's eqns become

$$\boxed{\partial^{[\mu} F^{\nu\rho]} = 0 \,, \quad \partial_\mu F^{\mu\nu} = 4\pi J^\nu}$$

completely antisymmetrize

sum on repeated indices

Lorentz force law a bit trickier:

Particle motion $\vec{x}(t) \Rightarrow \frac{dx^\mu}{dt} = (c, \vec{v})$

Define proper time τ by $\frac{d\tau}{dt} = \sqrt{1 - \frac{v^2}{c^2}}$,

then

$$\boxed{\underbrace{m \frac{d^2 x^\mu}{d\tau^2}}_{\text{4-force = m × 4-accel}} = q F^{\mu\alpha} \underbrace{\frac{dx_\alpha}{d\tau}}_{\text{4-velocity}}}$$

Lor. force law

- This formulation of electrodynamics is where we want to get to by end of course (spring semester).
- Key points:

(1) Fields carry energy etc so are "physical."

(2) Magnetic fields are a "relativistic effect."

(3) (Non-relativistic) experimental units (c, ϵ_0, μ_0) hide symmetries (Lorentz-invariance) of theory.

- General relativity is similar, but non-linear:

$\vec{g}(\vec{r},t) \longrightarrow g_{\mu\nu}(x) = g_{\nu\mu}(x)$ "metric tensor"

$$\boxed{D^\mu G_{\mu\nu} = 0 \quad , \quad G_{\mu\nu} = 8\pi G \cdot T_{\mu\nu}}$$ "Einstein eqns"

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}}$$ "geodesic eqn"

where

$G_{\mu\nu} \doteq$ "Einstein tensor" = nonlinear combo of 2^{nd} derivs of $g_{\mu\nu}$

$\Gamma^\mu_{\alpha\beta} \doteq$ "Levi-Civita connection" = non linear combo of 1^{st} derivs of $g_{\mu\nu}$

$T_{\mu\nu} \doteq$ "stress-energy tensor" = energy/momentum/pressure density

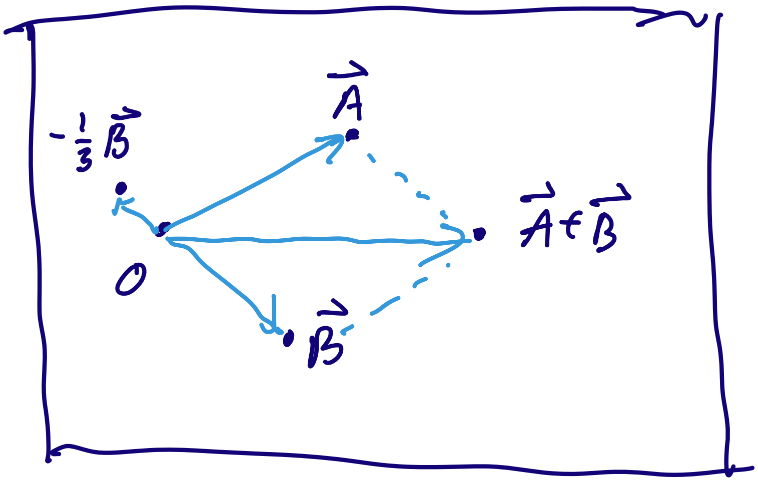
1. Vector calculus review

1.1 Linear algebra

- Vector space (real)

  "Flat" space $\sim \mathbb{R}^d$ ← dimension ≥ 0; $\mathbb{R}$ ← real line

  with a choice of origin:



$\vec{A}, \vec{B}, \ldots$ "vectors"

$$a(\vec{A}+\vec{B}) = a\vec{A} + a\vec{B} \qquad a \in \mathbb{R} \quad \text{"scalars"}$$

- Basis ordered set of d linearly indep't vectors

  $\{\vec{n}_1, \vec{n}_2, \ldots, \vec{n}_d\} \iff$ any vector can be written uniquely as linear combo

$$\vec{A} = A_1\vec{n}_1 + A_2\vec{n}_2 + \cdots + A_d\vec{n}_d = \sum_{i=1}^{d} A_i \vec{n}_i$$

$A_i \in \mathbb{R}$

- Dot (inner) product

  – Symmetric product to scalars:

$$\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} \in \mathbb{R}$$

- Bilinear:

$$(a\vec{A} + b\vec{B})\cdot\vec{C} = a\,\vec{A}\cdot\vec{C} + b\,\vec{B}\cdot\vec{C}$$
$$\vec{A}\cdot(b\vec{B} + c\vec{C}) = b\,\vec{A}\cdot\vec{B} + c\,\vec{A}\cdot\vec{C}$$

- In basis:

$$\vec{A}\cdot\vec{B} = \left(\sum_{i=1}^{d} A_i \vec{n}_i\right)\cdot\left(\sum_{j=1}^{d} B_j \vec{n}_j\right)$$
$$= \sum_{i,j=1}^{d} A_i B_j (\vec{n}_i\cdot\vec{n}_j)$$

So if know $d\times d$ symm. matrix of basis inner products, $\vec{n}_i\cdot\vec{n}_j$, then know any inner product.

- <u>Orthonormal basis</u> $\{\hat{x}_1, \dots, \hat{x}_d\}$ such that

$$\hat{x}_i\cdot\hat{x}_j = \delta_{ij}$$

$$\delta_{ij} \doteq \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$ "Kronecker delta"

- $\vec{A} = \sum_i A_i \hat{x}_i$ etc

$$\Rightarrow \vec{A}\cdot\vec{B} = \sum_{i,j} A_i B_j\, \hat{x}_i\cdot\hat{x}_j = \sum_{i,j} A_i B_j \delta_{ij}$$

$$= \sum_i A_i B_i$$

- $A \doteq \sqrt{\vec{A}\cdot\vec{A}} = \sqrt{\sum_i (A_i)^2}$ Pythagorean theorem

$\uparrow$ length of $\vec{A}$, ≥ 0.

$\Rightarrow \vec{A}\cdot\vec{B} = AB\cos\theta \leftarrow$ "angle between $\vec{A}$ & $\vec{B}$" (def'n)

$\vec{A}\cdot\vec{B} = 0 \iff \vec{A}$ orthogonal (perp.) to $\vec{B}$.

<u>3d vector space</u> $\simeq \mathbb{R}^3$

- o-n basis $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\} \doteq \{\hat{x}, \hat{y}, \hat{z}\}$

= "Cartesian basis"

- <u>Cross (exterior) product</u> (only in 3d)

$\vec{A}\times\vec{B}$ gives new <u>vector</u>, antisymmetric, bilinear:

$$\vec{C} = \vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$(a\vec{A}+b\vec{B})\times\vec{C} = a(\vec{A}\times\vec{C}) + b(\vec{B}\times\vec{C})$$

So if know cross products of a basis, can compute in general.

$$\boxed{\hat{x}\times\hat{y} \doteq \hat{z}, \quad \hat{y}\times\hat{z} \doteq \hat{x}, \quad \hat{z}\times\hat{x} \doteq \hat{y}}$$

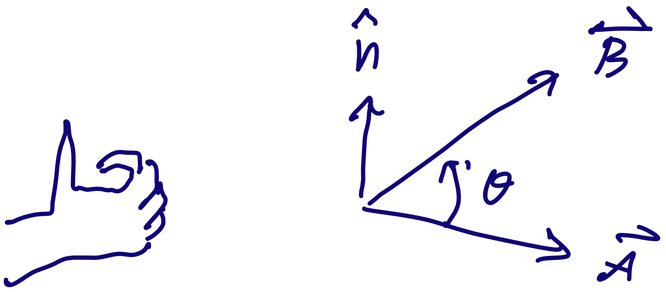
$\hat{x}\times\hat{x} = 0$ etc. $\qquad \hat{y}\times\hat{x} = -\hat{z}$, etc.

$$\Rightarrow \vec{A}\times\vec{B} = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$$

$$= \det \begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{pmatrix}.$$

$$\Rightarrow \vec{A}\times\vec{B} = AB\sin\theta \cdot \hat{n}$$

$\hat{n}$: unit vector $\perp$ to $\vec{A}$ & $\vec{B}$



θ, $\hat{n}$ defined by "right hand rule"

- Identities

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \det\begin{pmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{pmatrix} =$$ oriented volume of parallelopiped formed by $\vec{A}, \vec{B}, \vec{C}$

$$\Rightarrow \vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{C}\times\vec{A}) = \vec{C}\cdot(\vec{A}\times\vec{B})$$

$$= -\vec{A}\cdot(\vec{C}\times\vec{B}) \quad \ldots$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \qquad +(\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

$$(\vec{A}\times\vec{B})\times\vec{C} = -(\vec{B}\cdot\vec{C})\vec{A} + (\vec{A}\cdot\vec{C})\vec{B}$$

Not associative!

By using these identities, can always reduce to expressions with 0 or 1 x-product.

• <u>Position, displacement, and separation vectors</u>

– Position: $\vec{r} \doteq x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$

of point in 3-d with Cartesian coords (x,y,z).

$$r \doteq |\vec{r}| = \sqrt{x^2+y^2+z^2}.$$

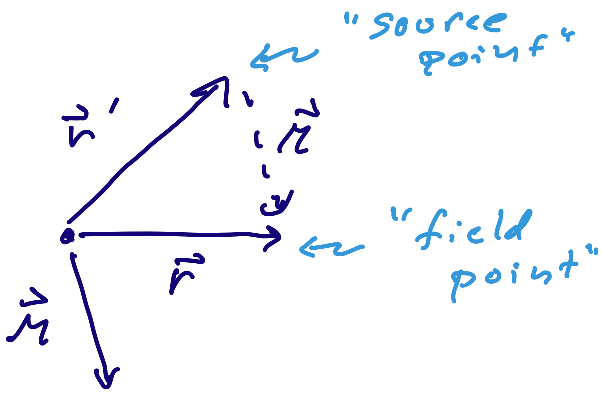
$$\hat{r} \doteq \frac{\vec{r}}{r} = \text{\underline{unit vector} pointing to } (x,y,z)$$

– Separation: 2 points with positions $\vec{r}, \vec{r}\,'$, separation is

$$\vec{\mathscr{r}} \doteq \vec{r} - \vec{r}\,'$$

$$\mathscr{r} \doteq |\vec{\mathscr{r}}|$$

$$\hat{\mathscr{r}} \doteq \frac{\vec{\mathscr{r}}}{\mathscr{r}}$$



– Infinitesimal displacement :

$$d\vec{r} \doteq d\vec{\ell} \doteq dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

is separation between 2 nearby points

$$\vec{r}\,' = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$\vec{r} = (x+dx)\,\hat{x} + (y+dy)\,\hat{y} + (z+dz)\,\hat{z}$$

• <u>Transformation of vector components under change of basis</u>

Basis : $\{ \vec{n}_i \;,\; i = 1 \cdots d \}$

Basis$'$: $\{ \vec{n}_i' \;,\; i = 1 \cdots d \}$

$$\vec{n}_i = \sum_j \vec{n}_j{}' R_{ji} \qquad \text{some } d \times d \text{ matrix } (R_{ij})$$

$$\vec{A} = \sum_i A_i \vec{n}_i = \sum_i A_i \Big(\sum_j R_{ji} \vec{n}_j{}' \Big)$$

$$= \sum_{ij} A_i R_{ji} \vec{n}_j{}' \doteq \sum_j A_j' \vec{n}_j{}'$$

$$\Rightarrow \boxed{A_j' = \sum_i R_{ji} A_i}$$

Turn into matrix expression :

$$\begin{pmatrix} A_1' \\ A_2' \\ \vdots \\ A_d' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1d} \\ R_{21} & R_{22} & \cdots & R_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ R_{d1} & R_{d2} & \cdots & R_{dd} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_d \end{pmatrix}$$

- If bases are o-n $\{\hat{x}_i\}$, $\{\hat{x}_i'\}$ then

$$\sum_j R_{ij} R_{kj} = \delta_{ik}$$

$$\Rightarrow \quad R R^T = I \quad \text{or} \quad R^{-1} = R^T$$

R = "orthogonal matrix".

- Rank-r tensor T is object with components

$$T_{i_1 i_2 \cdots i_r}$$

which transforms under c.o.b. as

$$\boxed{T'_{i_1 i_2 \cdots i_r} = \sum_{j_1 \cdots j_r} R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_r j_r} T_{j_1 \cdots j_r}}$$

Generalize: $\begin{cases} \text{scalars} = \text{rank } 0 \\ \text{vectors} = \text{rank } 1 \end{cases}$.

1.2 Differential calculus

- **Differentials** Given function $f: \mathbb{R}^d \to \mathbb{R}$

$$f(x_1 \cdots x_d) \in \mathbb{R}$$

$$\Rightarrow df \doteq \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_d} dx_d$$

$$\frac{\partial f}{\partial x_i} \doteq \left\{ \begin{array}{l} \text{derivative w.r.t. } x_i \text{, keeping} \\ \text{all other } x_j \text{ fixed} \end{array} \right\}$$

— $d \doteq$ "exterior derivative", "differential operator"

— d is linear: $d(af+bg) = a\,df + b\,dg$
($a, b \in \mathbb{R}$ constants, f, g functions)

— d is a derivation: "Leibniz rule"

$$d(fg) = f\,dg + g\,df$$

- **Gradient** Interpret

$$df \doteq (\vec{\nabla} f) \cdot d\vec{\ell}$$

$$\Rightarrow \boxed{\begin{array}{c} \vec{\nabla} f \doteq \dfrac{\partial f}{\partial x_1} \hat{x}_1 + \cdots + \dfrac{\partial f}{\partial x_d} \hat{x}_d \\ \text{in } \underline{\text{Cartesian}} \ \underline{\text{coordinates}} \text{ !} \end{array}}$$

← vector-valued function

- Geometrical interpretation:

$\vec{\nabla} f$ points in direction of maximum increase of f, and

$|\vec{\nabla} f|$ is slope of f along this direction

- Define gradient differential operator

$$\boxed{\vec{\nabla} \doteq \sum_{i=1}^{d} \hat{x}_i \frac{\partial}{\partial x_i} \qquad \text{(Cartesian coords!)}}$$

$\vec{\nabla}$: scalar functions $\longrightarrow$ vector-valued functions

- $f(x)$: scalar function

$\vec{v}(x)$: vector-valued function

Then can form:

Gradient: $\vec{\nabla} f$ $\in$ vector-valued func

Divergence: $\vec{\nabla} \cdot \vec{v}$ $\in$ scalar func

Curl: $\vec{\nabla} \times \vec{v}$ $\in$ vector-valued func

? : $\vec{\nabla} \vec{v}$ $\in$ tensor-valued func

- Divergence

$$\boxed{\vec{\nabla}\cdot\vec{v} = \sum_{j=1}^{d} \frac{\partial v_j}{\partial x_j} \quad \text{(Cartesian coords.)}}$$

$\vec{\nabla}\cdot\vec{v} \propto$ rate $\vec{v}$ "spreads out"

- Curl (d=3 only!)

$$\boxed{\vec{\nabla}\times\vec{v} = \det\begin{pmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{pmatrix} = \cdots \quad \text{(Cartesian coords.)}}$$

$|\vec{\nabla}\times\vec{v}| \propto$ "vorticity" of $\vec{v}$

- Product rules $\vec{\nabla}$ linear diff. op. $\Rightarrow$

$$\vec{\nabla}(af+bg) = a\vec{\nabla}f + b\vec{\nabla}g \qquad a,b\in\mathbb{R} \text{ const.}$$

$$\vec{\nabla}\cdot(a\vec{v}+b\vec{w}) = a\vec{\nabla}\cdot\vec{v} + b\vec{\nabla}\cdot\vec{w}$$

$$\vec{\nabla}\times(a\vec{v}+b\vec{w}) = a\vec{\nabla}\times\vec{v} + b\vec{\nabla}\times\vec{w}$$

$$\boxed{\begin{aligned} \vec{\nabla}(fg) &= f\vec{\nabla}g + g\vec{\nabla}f \\ \vec{\nabla}(\vec{v}\cdot\vec{w}) &= \vec{v}\times(\vec{\nabla}\times\vec{w}) + \vec{w}\times(\vec{\nabla}\times\vec{v}) \\ &\quad + (\vec{v}\cdot\vec{\nabla})\vec{w} + (\vec{w}\cdot\vec{\nabla})\vec{v} \end{aligned}}$$

$$\vec{\nabla}\cdot(f\vec{v}) = f\,\vec{\nabla}\cdot\vec{v} + \vec{v}\cdot\vec{\nabla}f$$

$$\vec{\nabla}\cdot(\vec{v}\times\vec{w}) = \vec{w}\cdot(\vec{\nabla}\times\vec{v}) - \vec{v}\cdot(\vec{\nabla}\times\vec{w})$$

$$\vec{\nabla}\times(f\vec{v}) = f\,\vec{\nabla}\times\vec{v} - \vec{v}\times\vec{\nabla}f$$

$$\vec{\nabla}\times(\vec{v}\times\vec{w}) = (\vec{w}\cdot\vec{\nabla})\vec{v} - (\vec{v}\cdot\vec{\nabla})\vec{w} - \vec{w}(\vec{\nabla}\cdot\vec{v}) + \vec{v}(\vec{\nabla}\cdot\vec{w})$$

useful!

– 2 derivatives:

$$\left.\begin{aligned}\vec{\nabla}\cdot\vec{\nabla}f &\doteq \nabla^2 f = \sum_{j=1}^{d}\frac{\partial^2 f}{\partial x_j^2}\\ \nabla^2\vec{v} &\doteq \sum_{j=1}^{d}(\nabla^2 v_j)\,\hat{x}_j\end{aligned}\right\}\ \text{Laplacian (cart. coords.)}$$

$$\vec{\nabla}\times\vec{\nabla}f = 0$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{v}) = 0$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{v}) = \underbrace{\vec{\nabla}(\vec{\nabla}\cdot\vec{v})}_{\text{gradient of divergence}} - \nabla^2\vec{v}$$